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Ganesh Manjhi and Meeta Keswani Mehra

National Institute of Public Finance and Policy
New Delhi

Dynamics of the Economics of Special Interest Politics[1]

Ganesh Manjhi[2] **and Meeta Keswani Mehra**[3]

Abstract

This paper derives the solution to differential games, when there are four sets of players, namely – two political parties (politicians), voters and a special interest group. The basic results are similar as Lambertini (2001, 2014). We find that, an open-loop equilibrium collapses to a closed-loop equilibrium. Therefore, the open-loop equilibrium is a sub-game perfect. Further, the private optimum is always higher than the social optimum in terms of the provision of the expenditure on public good. That is, if both the parties have access to public expenditure for the provision of the expenditure on public good they have the tendency to overspend and can incur higher deficits. Consequently, voters vote retrospectively to the party which overspend and results in higher fiscal deficits. Similarly, a larger private optimal regulatory benefit helps the political parties to receive higher financial contribution. Overall, the fiscal deficit in excess of certain level of threshold can create higher cost to the voters and hence the economy as the future tax and this is more so in the presence of special interest group.

Keywords: Special Interest Group, Public Good, Fiscal Deficit, Regulatory Benefit, Financial Contribution, Differential Games

JEL Classification Codes: C73, E6, H3

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[2]Assistant Professor, Gargi College, University of Delhi and Research Scholar, Centre for International Trade and Development, School of International Studies, Jawaharlal Nehru University, New Delhi - 110067. Email: ganeshtrx@gmail.com
[3]Professor of Economics, Centre for International Trade and Development, Jawaharlal Nehru University, New Delhi - 110067. Email: meetakm@mail.jnu.ac.in

1. Background

The democratic form of government was born around 5th to 4th century BC after the revolt by Harmodios and Aristogeiton and the family of Alkmaeonidai in Athens against the tyrant Hipparchos, who was then in the political power.[4] Since then, democracy has been evolving exuberantly. Today, one thinks it as the normative concept, as Pluto has said, 'it is stored up in heaven, but unhappily has not yet been communicated to us (Crick, 2002).' In fact, the understanding of the democracy can best be attributed as the positive concept and not normative as, 'all things bright and beautiful'. In short, the democracy of 'one man, one vote' is just the beginning of the route to democracy. Among many, some positive components of the concept of democracy can be ascribed to the relationship between the political parties and the special interest groups, where both of them mutually help each other such as, while in power the former provides the regulatory benefit to the latter and in return gets the financial contribution, particularly for the election campaign. In fact, the relationship between interest groups and the politician can be traced way back around 60 BC to 53 BC Roman Empire, when Julius Caesar was aiming for the power (consul of the Gaul in Roman Empire) and financially assisted by Marcus Licinius Crassus (Crassus, the wealthiest in Roman history) and Gnaeus Pompey Magnus.[5]

Today, the voting behaviour can be influenced by many different ways, and the existence of the interest group is just one of them. In fact, interest groups have become non- separable part of the democracy as also depicted by political scientists such as, Bentley (1908), Schattschneider (1935), Truman (1951), and more recently by economists, Olson Jr. (1971) and Stigler (1975), Austen-Smith (1987), Borooah & Ploeg (1983), Grossman & Helpman (1994, 1995a, 1995b, 1996, 1999, 2001), Goldberg & Maggi (1999) and Persson (1998). Interest groups can be of various forms such as environmental lobbies shouting slogans outside

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[4] The Athenian leader Cleisthenes introduced a system of political reforms that he called demokratia, or 'rule by the people' in the year 507 BC. The democracy in its totality is a normative concept, where everyone expects to exercise their rights and privileges, but today in reality; it is actually the positive concept at which none of the country into its democratic bliss. In conclusion, the democracy of 'one man one vote' is just the beginning of the route to democracy.

[5] The trio - Caesar, Crassus and Pompey formed a group famously known as 'the first triumvirate' and they ruled the Roman Empire for many years. Crassus is also considered as one of the wealthiest in the world history in general, and Roman Empire in particular. In return, according to Plutarch, both Crassus and Pompey got the tax break and land grants. In particular, Crassus accumulated ocean of wealth and power, like the vast sum of 7100 talents, owner of huge real estate and silver mines. He owned herd of slaves and having enormous wealth that he could fund his own army.

climate change summits, trade union forcing governments for higher minimum wage, farmers asking for higher Minimum Support Price (MSP) of paddy, group of developing countries voicing at WTO summits against the rich countries that provides higher agriculture subsidy to their farmers and so on. The relationship between politics and interest group in the democracy in the words of Kuttner is as follows:

> *The essence of political democracy - the franchise - has eroded, as voting and face-to-face politics give way to campaign-finance plutocracy… [T]here is a direct connection between the domination of politics by special interest money, paid attach ads, strategies driven by polling and focus groups - and the desertion of citizens … People conclude that politics is something that excludes them.*(Kuttner (1987) quoted in Caplan, (2008)).

While, closely looking at the positive concept of democratic electoral politics, where the special interest group intimately involves, there could be huge loss of revenue and unintended expenditure in the economy. Here, we take the dynamic approach of analyzing the relationship between political party/individual, interest group and voters. The basic results are as follows:

1. Open loop solution collapsed as the closed loop solution, therefore, open loop is a subgame perfect.
2. Private optimum is always higher than the social optimum in terms of the offer of the expenditure on public goods, regulatory benefit and received voting support and financial contribution.
3. Parties having access to public expenditure, the provision of the expenditure on the public good has a tendency to overspend and incur higher deficits. Consequently, voters vote retrospectively to the party which overspend and results in higher deficits. This phenomenon can have higher burden on voters as the future tax.
4. The private optimal regulatory benefit in return helps the political parties to receive higher financial contribution.

2. Relation with Literature

The last few decades cover the wide range of literature on the relationship between the interest groups and the political parties/politicians. This relationship has been modelled in many different ways, including variety of variables; however, the major link that explains this relationship is the financial contribution by the interest groups to the political parties/politicians in return of the regulatory benefit. Some of the major contributions such as - Grossman & Helpman (1994, 1995a, 1995b, 1996, 1999, 2001) and Goldberg & Maggi (1999) look at the quid pro quo relationship between the special interest group and political parties (politicians). In these models, the basic idea is that the interest groups provide financial contribution to the political parties/politicians and in return they want the economic policies to be positively biased to them. However, voters can reject this rent-seeking relationship between political parties and interest groups, however, this rejection can be well substituted by the ideology that voters subscribe to. Bennedsen & Feldmann (2006) state that the interest groups offer contribution to politicians to get favours in the policy decisions whereas interest groups contribute to influence the electoral outcome rather than influencing candidates policy choices directly (Magee, 2007). Potters & Winden (1992) and Potters, Sloof, & Van Winden (1997) both, model the financial contribution and lobbying for information where, the latter work extends the campaign contribution model of politicians based on the contributions by the interest groups. They also find that interest groups contribute to the candidate's campaign rather than direct endorsements. Denzau & Munger (1986), Mitchell & Munger (1991) and Lohmann (1995) find that, if the interests of the interest group aligned with that of the policymaker's constituency and voters are neutral over the policies, they have costless access and report their information truthfully, whereas, if there exist voters' preference over policy in effect, then interest group has to pay a higher price to stay relevant. Wittman (2007) shows that the presence of interest groups is welfare improving if they endorse good quality leaders in the presence of uninformed leaders, whereas, the interaction of interest group, voters and government create electoral cycles through expenditure composition and exchange rate (Bonomo & Terra, 2010). Lohmann (1998) finds that, the political decisions are often biased in favour of special interest at the cost of mass voters, and they are frequently inefficient. That is, the losses incurred by the majority exceeds the gains enjoyed by few minority, whereas, the organized groups with true preference over policy and contribution to the government, internalize the marginal cost in the government policy decisions in terms of the allocation of the public good. Also, the organized interest groups get more than the social optimum (Persson, 1998). Another extreme, if buying of votes by interest groups is allowed, voters may allow policy to deviate somewhat from their ideal point to prevent excessive vote buying (Snyder & Ting, 2008).

Some papers based on competition between political parties or competitions between interestgroups are in order. Borooah & Ploeg (1983), Coughlin, Mueller, & Murrell (1990a) in an electoral competition model with special interest group find that, the political parties have equilibrium strategies that can be viewed as maximizing social objective function. The strength of the interest group is seen as the politician's perception of a group's reliability in delivering the votes for its members. Coate (2004) finds, the policy-motivated parties compete by selecting candidates, and interest groups provide contributions to enhance the electoral prospects of like-minded candidates, and contributions are used to finance advertising campaigns that provide voters with information about candidate's ideology. Prat (2002), Gavious & Mizrahi (2002) and Epstein & O'Halloran (1995) state, well prior to the elections the politicians in office should invest constant level of resources, while for a certain period close to the elections, the politician increases or decreases investment depending on the electoral significance of that interest group. By extending the work to several interest groups they find that, at each point in time, the politician should invest in that group that contribute the most for his or her political interest, whereas, the interest groups can avoid contributing if the ideological stand of the decision maker has too much uncertainty (Martimort & Semenov, 2007; Martimort & Semenov, 2008).

Some analytical cum empirical papers are in order. Bouton, Conconi, Pino, & Zanardi (2013) in the concept of the 'paradox of gun' find that, despite 90% of the citizen support the regulation on the open purchase of guns in US, they fail in the senate. In fact, senators closer to the election, are more likely to vote for pro-gun and this is true in the presence of financial contribution to the senators by gun lobbies and even without it. Further, Bouton, Conconi, Pino, & Zanardi (2014) state that, voters vote on the basis of primary and secondary policy issues, where the former can basically attract the citizen voters through public expenditure and the latter might mean to gun control. Goss (2010) explains this as - the gun lobbies in US are intense, well organized and are willing to vote for and against the candidates purely on the basis of their position on gun control. They are 'highly motivated', 'intense minority', who prevail over a 'relatively apathetic majority'. Welch (1980) designed the contribution function of the 7 interest groups and tested1974 candidates for US and found that interest groups contribute to get the political favour, not to affect the electoral outcome. Keiser & Jones Jr. (1986) find that the influence of the American Medical Association (AMA) campaign contributions produced more results over a series of decisions but, the legislators who failed to return the AMA's favour on one item might help out in another measures. In the empirical exercise by Huber & Kirchler (2013), the companies experience abnormal positive post-election returns, which have higher percentage of contributions to the eventual winner in the Unites States from 1992 to 2004. In the context of India, Kapur & Vaishnav (2013) show that, politicians and builders engage in a quid pro quo, whereby the former position their illicit assets with the latter and the latter rely on the former for favourable delivery of the wealth during the election. Sadiraj, Tuinstra, & Van Winden (2010) find identification of voters with interest groups improves the electoral chances of the challenger whereas, Fiorino & Ricciuti (2009) find that government spending was sensitive to the preferences of heavy industry rather than those of textile and cereal cultivators during 1876 to 1913 Italy. However, mixed results cannot be denied for some cases, such as, Etzioni (1985) finds interest group as the threat to the pluralist democracy in the citizen's view, but the conventional wisdom of the political science find it beneficial. In fact, the elimination of the interest group is not possible and rather competing interest groups will curb each other.

The current work enrich the aforementioned literature of the interaction between political parties/politicians and interest groups. The important value additions are - the private optimum is always higher than the social optimum in terms of voting support, offer and actual expenditure on public good provision, financial contribution and regulatory benefit, in fact, it is more so with the linear cost structure in the objective function. This coincides with the results by Lohmann (1998), (Persson, 1998) and Lambertini (2001, 2014). In the presence of voters 'fiscal illusion', this interactive mechanism will work well otherwise, the quid pro quo relationship might lead to unequal society and plutocracy; ignoring the majority.

3. Model Set Up and Assumptions

This paper focuses on the relationship between two political parties/individual contesting election, voters and a special interest group. A priory, the political parties offer expenditure on public goods provision (this assumption has been relaxed later into actual expenditure), voters observe the offer, and vote retrospectively. In addition, contestants also offer regulatory benefit (assumption later relaxed as the actual regulatory benefit) to the industrial interest/lobby group in exchange of the financial contribution to meet the large expenses of the election ads and campaigns.

Assume that the economy is on balance budget or runs at some sustainable and acceptable level of deficit at $\hat{d}$. If the functioning of the economy exceeds the sustainable level of deficit $d(t) > \hat{d}$ then, the cost of the provision of public goods rise. The cost to the economy as a function of the government expenditure can be depicted as $C_1(t) = \frac{\varphi_1}{2}\frac{g_i(t)^2}{\tau}$ in the model. The $g_i(t)$ is the offer of the expenditure on public good by two political parties, i = 1, 2 where, both the parties have the access to public expenditure. The voters vote retrospectively to the parties based on promise to deliver the public goods. The voting support by the citizen voters are depicted as $m_i(t)$, where i = 1, 2.

There exists an industrial special interest group (SIG), which is powerful enough to affect the economy. The industrial special interest group offer financial contribution to the two political parties, which tend to contest election. The political contestants offer the regulatory benefit to the SIG and the costs of the regulatory benefit can be depicted as:$C_1(t) = \frac{\varphi_2}{2} r_i(t)^2$, where, $r_i(t)$ is the regulatory benefit to the SIG. In return, the political parties (politicians) can receive the financial contribution $b_i(t)$, from a single SIG, where i = 1, 2. So, apart from financial contribution, SIG vote to the preferred political party.

The total cost to the economy can be depicted as the gross deficit, which is the sum total of the fiscal deficit due to expenditure on public good and the lost revenue due to the regulatory benefit to SIG (and financial contribution in return). The gross cost to the economy at the time period 't' is, $D(t) = C_1(t) + C_2(t)$. Clearly, this shows that the economy is functioning at $D(t) > \hat{d}$ in the presence of special interest group. In reality, the economy wide fiscal deficit is explained as the function of expenditure itself and not the transactions in the dark (transactions between interest groups and political parties). The election will take place with certainty at date T and
$t \in [0, T]$, where 't' is any of the date in the electoral period. The political parties/politicians offer expenditure , $g_i(t)$ on the provision of public good in the economy and regulatory benefit, $r_i(t)$ to the special interest group. At the terminal date T, voters and special interest group vote for the party/individual, that they prefer to vote. Here, we extend the model by Lambertini (2001, 2014) by including the SIG. To find out the optimal solution we use the method of optimal control (Chiang, 1992; Long, 2010).

4. Model – I

The dynamic behaviour of the voting support function can be following:

$$\dot{m}_i(t) = g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t) \tag{1}$$

The dynamics of the financial contribution equation given by the SIG is:

$$\dot{b}_i(t) = r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t) \tag{2}$$

The utility maximization function of the two political parties can take the following form:

$$\underset{\{g_i, r_i\}}{Max} \int_0^T \left\{\theta\left[\delta_i m_i(t) - \frac{\varphi_1}{2}\frac{(g_i(t))^2}{\tau}\right] + (1-\theta)\left[\gamma_i b_i(t) - \frac{\varphi_2}{2}(r_i(t))^2\right]\right\} e^{-\rho t} dt +$$
$$e^{-\rho T} S_1[m_i(T)] + e^{-\rho T} S_2[b_i(T)] \tag{3}$$

where, θ is the relative taste parameter of the political contestants and $0 < \theta < 1$. If $\theta > \frac{1}{2}$, then political party care more about the net voting support and less about the net gain from the SIG's financial contribution, and opposite is true, if $\theta < \frac{1}{2}$. If $\theta = 1$, then model will collapse to the framework of Lambertini (2001, 2014) and utility maximization exercise will be without the SIG. The parameter φ_1 is the cost attached to the government expenditure on public good which might be at $\hat{d}$ or exceeds to it, and φ_2 is the cost attached because of regulatory benefit $r_i(t)$ to the SIG. The $\delta_i m_i(t)$ is the gross benefit to political parties because of the voting support and $\gamma_i b_i(t)$ is the gross benefit out of financial contribution from the SIG. Also the model is not the full information model where voters know everything about the relationship between the political parties and the SIG. The variable τ is lump-sum tax and even if this is equal to the government expenditure ($g_i(t)$ at date 't', cost is still attached to the economy in terms of interest payments to the past debts. The two control variables in this case are $g_i(t)$ and $r_i(t)$ and their respective state variables are $m_i(t)$ and $b_i(t)$.

When the voting pattern and financial contribution pattern are according to the equation (1) and (2), the corresponding closed-loop Hamiltonian for party (or candidate) 'i' is-

$$\mathcal{H}_i(t) = e^{-\rho t}\left[\theta\left[\delta_i m_i(t) - \frac{\varphi_1}{2}\frac{g_i(t)^2}{\tau}\right] + (1-\theta)\left[\gamma_i b_i(t) - \frac{\varphi_2}{2} r_i(t)^2\right] + \lambda_{ii}(t)\big[g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t)\big] + \lambda_{ij}(t)\big[g_j(t) - \alpha_1 g_i(t) - \alpha_2 m_j(t) + \alpha_3 b_j(t)\big] + \psi_{ii}(t)\big[r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t)\big] + \psi_{ij}(t)\big[r_j(t) - \beta_1 r_i(t) - \beta_2 b_j(t)\big]\right] \quad (4)$$

Where, $\lambda_{ii}(t) = \mu_{ii}(t)e^{\rho t}$, and $\mu_{ii}(t)$ is the co-state variable associated with $m_i(t)$ and $\psi_{ii}(t) = \eta_{ii}(t)e^{\rho t}$ and $\eta_{ii}(t)$ is the co-state variable associated to $b_i(t)$.

4.1 The private optimum

We investigate the outcome of the non-cooperative game where each party maximizes its own discounted (constrained) utility. On the basis of equation (4), the following holds.

***Lemma 1*:** Theopen-loop equilibrium is collapsed as the closed-loop equilibrium. Therefore, the open-loop equilibrium is a sub-game perfect.

Proof: The first order conditions are:

$$\frac{\partial \mathcal{H}_i(t)}{\partial g_i(t)} = -\frac{\theta\varphi_1}{\tau} g_i(t) + \lambda_{ii}(t) - \alpha_1\lambda_{ij}(t) = 0 \quad (5)$$

$$\text{and, } \dot{\lambda}_{ii}(t) - \rho\lambda_{ii}(t) = -\frac{\partial \mathcal{H}_i(t)}{\partial m_i(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial g_i(t)}\frac{\partial g^*(t)}{\partial m_i(t)}, \forall i,j \quad (6)$$

Similarly,

$$\frac{\partial \mathcal{H}_i(t)}{\partial r_i(t)} = -(1-\theta)\varphi_2 r_i(t) + \psi_{ii}(t) - \beta_1\psi_{ij}(t) = 0 \quad (7)$$

$$\text{and, } \dot{\psi}_{ii}(t) - \rho\psi_{ii}(t) = -\frac{\partial \mathcal{H}_i(t)}{\partial b_i(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial r_i(t)}\frac{\partial r^*(t)}{\partial b_i(t)}, \forall i,j \quad (8)$$

The initial conditions are $m_i(0) = m_{i0}$ and $r_i(0) = r_{i0}$ and the respective scrap value functions (SVF) are $\lambda_{ii}(T) = 0$ and $\psi_{ii}(T) = 0$, which further respectively imply that, $\lambda_{ii}(T)S[m_i(T)] = 0$ and $\psi_{ii}(T)S[b_i(T)] = 0$. Now, to see the dynamic relationship between control and state variables, the solution with respect to $g_i(t)$ gives, $g_i^* = \frac{\tau}{\theta\varphi_1}\big[\lambda_{ii}(t) - \alpha_1\lambda_{ij}(t)\big]$, which obtains $\frac{\partial g^*(t)}{\partial m_i(t)} = 0$ and $\frac{\partial g^*(t)}{\partial b_i(t)} = 0$. This implies that, $\lambda_{ij}(t) = 0$ is admissible and accordingly has the open-loop solution and it is a degenerate close-loop solution. Similarly, the solution with respect to $r_i(t)$ gives, $r_i^* = \frac{1}{\varphi_2(1-\theta)}\left[\psi_{ii}(t) - \beta_1\psi_{ij}(t)\right]$, which further gives the result as, $\frac{\partial r^*(t)}{\partial m_i(t)} = 0$ and $\frac{\partial r^*(t)}{\partial b_i(t)} = 0$. This shows that, $\psi_{ij}(t) = 0$ is admissible and accordingly the open-loop solution is a degenerate close-loop solution.

The open loop solution leads to the following result:

Proposition 1: *At the open-loop steady state equilibrium, party* i's *offer of the expenditure on public good is,* $g_i^* = \Omega_1\delta_i$ *and regulatory benefit,* $r_i^* = [\Omega_2\gamma_i + \Omega_2\Omega_3\delta_i]$. *The respective steady state voting support,* $m_i^* = \frac{1}{\alpha_2}\left[\Omega_2\frac{\alpha_3}{\beta_2}\left(\gamma_i - \alpha_1\gamma_j\right) + \left(\Omega_1 + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\right)\delta_i - \left(\Omega_1\alpha_1 + \Omega_2\Omega_3\frac{\alpha_3\beta_1}{\beta_2}\right)\delta_j\right]$ *and financial contribution is,* $b_i^* = \frac{1}{\beta_2}\left[\Omega_2\left(\gamma_i - \beta_1\gamma_j\right) + \Omega_2\Omega_3\left(\delta_i - \beta_1\delta_j\right)\right]$. *The party (politician) i's steady state offers of the public good and regulatory benefit given, will be higher than j's if the respective* $\delta_i > \delta_j$ and $\gamma_i > \gamma_j$. *This also ensures that voting support and financial contributions received by i's will always be higher than party j's if* $\delta_i > \delta_j$ and $\gamma_i > \gamma_j$. [Where, $\Omega_1 = \frac{\tau}{\varphi_1(\rho+\alpha_2)}$, $\Omega_2 = \frac{1}{[\varphi_2(\rho+\beta_2)]}$, $\Omega_3 = \frac{\theta\alpha_3}{[(1-\theta)(\rho+\alpha_2)]}$]

Proof: See Appendix

Proposition 2: The open-loop equilibrium $\{g_i^*, r_i^*, m_i^*, b_i^*\}$ is saddle point equilibrium.

Proof: See Appendix

4.2 The Social Optimum

If there exists a benevolent social planner who chooses the vector of the offer of expenditure on public good $g_i(t)$ and the regulatory benefit $r_i(t)$. The planner maximize the collective welfare defined as the sum total of the parties discounted pay-off under the constraint eq. (1) and eq. (2).

The Hamiltonian function is:

$$\mathcal{H}_i^{SO}(t) = e^{-\rho t}\left[\theta\left[\delta_i m_i(t) + \delta_j m_j(t) - \frac{\varphi_1}{2}\frac{(g_i(t))^2}{\tau} - \frac{\varphi_1}{2}\frac{(g_j(t))^2}{\tau}\right] + (1-\theta)\left[\gamma_i b_i(t) + \gamma_j b_j(t) - \frac{\varphi_2}{2}(r_i(t))^2 - \frac{\varphi_2}{2}(r_j(t))^2\right] + \lambda_i(t)\left[g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t)\right] + \lambda_j(t)\left[g_j(t) - \right.\right.$$

$\alpha_1 g_i(t) - \alpha_2 m_j(t) + \alpha_3 b_j(t)\big] + \psi_i(t)\big[r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t)\big] + \psi_j(t)\big[r_j(t) - \beta_1 r_i(t) -$

$$\beta_2 b_j(t)\big]\Big] \tag{9}$$

The social optimum exercise reduced to the following proposition –

***Proposition 3*:** At the social optimum, party *i*'s offer of the expenditure on public good and regulatory benefit are as follows -- $g_i^{so} = \Omega_1[\delta_i - \alpha_1\delta_j]$ and $r_i^{so} = \big[\Omega_2(\gamma_i - \beta_1\gamma_j) + \Omega_2\Omega_3(\delta_i - \beta_1\delta_j)\big]$. The respective voting support from the citizen voters and the financial contribution by the SIG are - $m_i^{so} = \frac{1}{\alpha_2}\Big[\Omega_2\frac{\alpha_3}{\beta_2}\{(1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j\} + \Omega_4\delta_i - 2\Omega_5\delta_j\Big]$ and $b_i^{so} = \frac{1}{\beta_2}\Big[\Omega_2\{(1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j\} + \Omega_2\Omega_3\{(1+\beta_1^2)\delta_i - 2\beta_1\delta_j)\}\Big]$. [Where, $\Omega_1 = \frac{\tau}{[\varphi_1(\rho+\alpha_2)]}$, $\Omega_2 = \frac{1}{[\varphi_2(\rho+\beta_2)]}$, $\Omega_3 = \frac{\theta\alpha_3}{[(1-\theta)(\rho+\alpha_2)]}$, $\Omega_4 = \Big[\Omega_1(1+\alpha_1^2) + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}(1+\beta_1^2)\Big]$ and $\Omega_5 = \Big[\Omega_1\alpha_1 + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\Big]$.]

Proof: See Appendix

***Proposition 4*:** Thesolution for two control variables – expenditure on public good and regulatory benefit to the industrial lobby (SIG) and the state variables – voting support and financial contributions are the saddle point equilibrium.

Proof: This is analogous to the solution for the private optimum, hence skipped.

4.3 Private vs. Social Optimum

4.3.1 Hypothetical Case

This is the case when two political parties offer the expenditure on public good- $g_i(t)$ and $g_j(t)$ and the regulatory benefits of $r_i(t)$ and $r_j(t)$. In return, both parties get the voting support of $m_i(t)$ and $m_j(t)$ from voters and financial contribution of $b_i(t)$ and $b_j(t)$ from special interest groups. In this hypothetical case where every player is contingent to the promise, the solutions for the private and social optimums are as follows:

$$G^* = g_i^* + g_j^* = \Omega_1(\delta_i + \delta_j) \tag{10}$$

$$G^{so} = g_i^{so} + g_j^{so} = \Omega_1(1-\alpha_1)(\delta_i + \delta_j) = (1-\alpha_1)G^* \tag{11}$$

$$R^* = r_i^* + r_j^* = \Omega_2(\gamma_i + \gamma_j) + \Omega_2\Omega_3(\delta_i + \delta_j) \quad (12)$$

$$R^{so} = r_i^{so} + r_j^{so} = \Omega_2(1-\beta_1)(\gamma_i + \gamma_j) + \Omega_2\Omega_3(1-\beta_1)(\delta_i + \delta_j) = (1-\beta_1)R^* \quad (13)$$

$$M^* = m_i^* + m_j^* = \frac{1}{\alpha_2}\left[\left[\Omega_2\frac{\alpha_3}{\beta_2}(1-\beta_1)\right](\gamma_i + \gamma_j) + \left[\Omega_1(1-\alpha_1) + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}(1-\beta_1)\right](\delta_i + \delta_j)\right] \quad (14)$$

$$M^{so} = m_i^{so} + m_j^{so} = \frac{1}{\alpha_2}\left[\left[\Omega_2\frac{\alpha_3}{\beta_2}(1-\beta_1)^2\right](\gamma_i + \gamma_j) + \left[\Omega_1(1-\alpha_1)^2 + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}(1-\beta_1)^2\right](\delta_i + \delta_j)\right] \quad (15)$$

$$B^* = b_i^* + b_j^* = \frac{1}{\beta_2}\left[\Omega_2(1-\beta_1)(\gamma_i + \gamma_j) + [\Omega_2\Omega_3(1-\beta_1)](\delta_i + \delta_j)\right] \quad (16)$$

$$B^{so} = b_i^{so} + b_j^{so} = \frac{1}{\beta_2}\left[[\Omega_2(1-\beta_1)^2](\gamma_i + \gamma_j) + [\Omega_2\Omega_3(1-\beta_1)^2](\delta_i + \delta_j)\right] = (1-\beta_1)B^* \quad (17)$$

where, $0 < \alpha_1, \beta_1 < 1$. Comparing eq. (10) with eq. (11) and eq. (12) with eq. (13) observe that, the private optimum is higher than the social optimum for both, the offer of the expenditure on public good in the economy and regulatory benefit provided to the SIG. Similarly, comparing eq. (14) with eq. (15) and eq. (16) with eq. (17) we find that, private optimum of voting support and financial contributions are higher than the respective social optimums. Accordingly the optimum private gross loss (total deficits) to the economy is higher than the social optimum. That is, $D^* > D^{so}$.

4.3.1 Actual Execution of the Model

In reality, in an electoral period [0, *T*], there is only one party who is in power, known as the incumbent. Consider, party '*i*' is the incumbent and '*j*' is the second biggest party who is in opposition in the parliamentary democracy. So, in the electoral period [0, T], g_i can be actual expenditure on public good in the economy, whereas,g_j is the offer of public expenditure by the opponent. In that case, $g_i > g_j$ is always true. However, if the opposition is competitive enough in the economy then, opponent will always offer $g_j \geq g_i$. In reality, the possible scenario at the date of terminal time '*T*', g_i is the actual expenditure incurred by incumbent whereas, g_j is the competitive offer of expenditure by the opponents. Thus, the actual expenditure incurred by the incumbent is $g_i >> 0$ and $g_j = 0$. Considering the above explanation, the private and social optimum has been observed as-

$$G^* = g_i^* + g_j^* = \Omega_1 \delta_i \tag{18}$$

$$G^{so} = g_i^{so} + g_j^{so} = \Omega_1(1-\alpha_1)\delta_i = (1-\alpha_1)G^* \tag{19}$$

The φ_1 associated with g_j will be zero because g_j itself is zero. Accordingly, Ω_1 associated with g_j^* will be zero. Thus, for g_j^* and $g_j^{so}$and $0 < \alpha_1 < 1$ we find, and $G^* > G^{so}$ is still true. The regulatory benefit is actually been given by the incumbent in the electoral period but the opponent can only offer the regulatory benefit and cannot really execute. So, in this case $r_i >> 0$ and $r_i = 0$. However, the financial contribution by the corporate lobby can be enjoyed by both the parties. The private and social optimum solution in the case of regulatory benefit is as follows:

$$R^* = r_i^* + r_j^* = \Omega_2\gamma_i + \Omega_2\Omega_3\delta_i \tag{20}$$

$$R^{so} = r_i^{so} + r_j^{so} = \Omega_2(1-\beta_1)\gamma_i + \Omega_2\Omega_3(1-\beta_1)\delta_i = (1-\beta_1)R^* \tag{21}$$

Similarly, voting support will be garnered by both, the incumbent and the opponent based on the actual expenditure incurred and the offer of the expenditure on public good is-

$$M^* = m_i^* + m_j^* = \frac{1}{\alpha_2}\left[\left[\Omega_2\frac{\alpha_3}{\beta_2}(1-\beta_1)\right]\gamma_i + \left[\Omega_1(1-\alpha_1) + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}(1-\beta_1)\right]\delta_i\right] \tag{22}$$

$$M^{so} = m_i^{so} + m_j^{so} = \frac{1}{\alpha_2}\left[\left[\Omega_2\frac{\alpha_3}{\beta_2}(1-\beta_1)^2\right]\gamma_i + \left[(\Omega_4 - 2\Omega_3)\delta_i - 2\Omega_3\delta_j\right]\right] \tag{23}$$

Even in this case, the voting support is, $M^* > M^{so}$.

Finally, the financial contribution given to the political parties '*i*' and '*j*' are as follows. These contributions are based on the actual regulatory benefit given by the incumbent and the offer of the regulatory benefit by the opponent. The private and social optimum solutions are as follows:

$$B^* = b_i^* + b_j^* = \frac{(1-\beta_1)}{\beta_2}[\Omega_2\gamma_i + \Omega_2\Omega_3\delta_i] \tag{24}$$

$$\mathrm{B^{so}} = \mathrm{b_i^{so}} + \mathrm{b_j^{so}} = \frac{(1-\beta_1)^2}{\beta_2}[\Omega_2\gamma_i + \Omega_2\Omega_3\delta_i] = (1-\beta_1)\mathrm{B}^* \quad (25)$$

Again, we find that, $B^* > B^{so}$. So, by relaxing the assumptions of the nature of two political parties from access to both, the public expenditure and regulatory power to only incumbent's access, the basic result does not change. That is, private optimum is still higher than the social optimum.

5. Model - II

If the dynamic behaviour of the voting support equation takes the following form:

$$\dot{m}_i(t) = \sqrt{g_i(t) - \alpha_1 g_j(t)} - \alpha_2 m_i(t) + \alpha_3 b_i(t) \quad (26)$$

And the dynamic equation of the financial contribution given by the SIG to two political parties is,

$$\dot{b}_i(t) = \sqrt{r_i(t) - \beta_1 r_j(t)} - \beta_2 b_i(t) \quad (27)$$

The utility maximization function of the two political parties can take the following form:

$$\underset{\{g_i, r_i\}}{Max} \int_0^T \left\{\theta\left[\delta_i m_i(t) - \varphi_1 \frac{g_i(t)}{\tau}\right] + (1-\theta)[\gamma_i b_i(t) - \varphi_2 r_i(t)]\right\} e^{-\rho t} dt + e^{-\rho T} S_1[m_i(T)] + e^{-\rho T} S_2[b_i(T)] \quad (28)$$

where, $0 < \theta < 1$, $0 < \alpha_1 < 1$ and $0 < \beta_1 < 1$

Lemma 2: Theopen-loop equilibrium is collapsed as the closed-loop equilibrium. Therefore, the open-loop equilibrium is a sub-game perfect.

Proof: The closed loop Hamiltonian of party '*i*' is,

$$\mathcal{H}_i(t) = e^{-\rho t}\Big\{\theta\left[\delta_i m_i(t) - \varphi_1 \frac{g_i(t)}{\tau}\right] + (1-\theta)[\gamma_i b_i(t) - \varphi_2 r_i(t)] + \lambda_{ii}(t)\left[\sqrt{g_i(t) - \alpha_1 g_j(t)} - \alpha_2 m_i(t) + \alpha_3 b_i(t)\right] + \lambda_{ij}(t)\left[\sqrt{g_j(t) - \alpha_1 g_i(t)} - \alpha_2 m_j(t) + \alpha_3 b_j(t)\right] + \psi_{ii}(t)\left[\sqrt{r_i(t) - \beta_1 r_j(t)} - \beta_2 b_i(t)\right] + \psi_{ij}(t)\left[\sqrt{r_j(t) - \beta_1 r_i(t)} - \beta_2 b_j(t)\right]\Big\} \quad (29)$$

Where, $\lambda_{ii}(t) = \mu_{ii}(t)e^{\rho t}$, and $\mu_{ii}(t)$ is the co-state variable associated with $m_i(t)$ and , $\psi_{ii}(t) = \eta_{ii}(t)e^{\rho t}$ and $\eta_{ii}(t)$ is the co-state variable associated to $b_i(t)$. The discounted scrap value function of the state variables are $e^{-\rho T}S_1[m_i(T)]$ and $e^{-\rho T}S_2[b_i(T)]$ respectively associated with $m_i(t)$ and $b_i(t)$.

The first order conditions from the Hamiltonian function of (29) are,

$$\frac{\partial \mathcal{H}_i(t)}{\partial g_i(t)} = -\frac{\theta\varphi_1}{\tau} + \frac{\lambda_{ii}(t)}{2\sqrt{g_i(t)-\alpha_1 g_j(t)}} - \frac{\alpha_1\lambda_{ij}(t)}{2\sqrt{g_i(t)-\alpha_1 g_j(t)}} = 0 \tag{30}$$

Similarly,

$$\frac{\partial \mathcal{H}_i(t)}{\partial r_i(t)} = -(1-\theta)\varphi_2 + \frac{\psi_{ii}(t)}{2\sqrt{r_i(t)-\beta_1 r_j(t)}} - \frac{\beta_1\psi_{ij}(t)}{2\sqrt{r_i(t)-\beta_1 r_j(t)}} = 0 \tag{31}$$

Plus the initial conditions are $m_i(0) = m_{i0}$ and $r_i(0) = r_{i0}$ and the respective TVC are $\lambda_{ii}(T) = 0$ and $\psi_{ii}(T) = 0$, which further respectively imply that, $\lambda_{ii}(T)S_1[m_i(T)] = 0$ and $\psi_{ii}(T)S_2[b_i(T)] = 0$. From eq. (30) it is immediate to check and establish, $\frac{\partial^2\mathcal{H}_i(t)}{\partial g_i(t)\partial m_i(t)} = 0$ and $\frac{\partial^2\mathcal{H}_i(t)}{\partial g_i(t)\partial b_i(t)} = 0\forall i,j$ and from (31) we get, $\frac{\partial^2\mathcal{H}_i(t)}{\partial r_i(t)\partial b_i(t)} = 0$ and $\frac{\partial^2\mathcal{H}_i(t)}{\partial r_i(t)\partial m_i(t)} = 0\forall i,j$. This shows that, $\lambda_{ij}(t) = 0$ and $\psi_{ij}(t) = 0$ is admissible and accordingly the open-loop solution is degenerate close-loop solution. This proves the Lemma 2. This proves Lemma 2. Lemma 2 state that, as the open loop equilibrium is strongly time consistent, it is sub-game perfect. That is, as in case of Model-I, the game is perfect or state redundant. In the remainder of the section in order to concentrate on the comparison between the private and social optima, we assume, $\delta_i = \delta$ and $\gamma_i = \gamma$. This entails that the model can only produce symmetric equilibria and therefore cannot determine which party (or candidate) ultimately wins the elections.

5.1 Private Optimum

When the voting support by citizen voters and the financial contribution pattern by the SIG are according to the equation (26) and (27), the relevant closed-loop Hamiltonian for party (politician) '*i*' is the following.

The relevant open loop Hamiltonian for party (politician) '*i*' can be re-written as follows-

$$\mathcal{H}_i(t) = e^{-\rho t}\Big\{\theta\left[\delta_i m_i(t) - \varphi_1\frac{g_i(t)}{\tau}\right] + (1-\theta)[\gamma_i b_i(t) - \varphi_2 r_i(t)] +$$
$$\lambda_i(t)\left[\sqrt{g_i(t)-\alpha_1 g_j(t)} - \alpha_2 m_i(t) + \alpha_3 b_i(t)\right] + \psi_i(t)\left[\sqrt{r_i(t)-\beta_1 r_j(t)} - \beta_2 b_i(t)\right]\Big\} \tag{32}$$

The open loop solution leads to following result:

Proposition 5: *At the open-loop steady state equilibrium, party 'i' can have expenditure on public good is,* $g_i^* = \frac{1}{4(1-\alpha_1)}[\Omega_1\delta]^2$ *and regulatory benefit,* $r_i{}^* = \frac{1}{4(1-\beta_1)}[\Omega_2\gamma + \Omega_2\Omega_3\delta]^2$. *The respective steady state voting support,* $m_i^* = \frac{1}{2\alpha_2}\left[\left(\Omega_2\frac{\alpha_3}{\beta_2}\right)\gamma + \left(\Omega_1 + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\right)\delta\right]$ *and financial contribution is,* $b_i^* = \frac{1}{2\beta_2}[\Omega_2\gamma + \Omega_2\Omega_3\delta]$. [Where, $\Omega_1 = \frac{\tau}{\varphi_1(\rho+\alpha_2)}$, $\Omega_2 = \frac{1}{[\varphi_2(\rho+\beta_2)]}$, $\Omega_3 = \frac{\theta\alpha_3}{[(1-\theta)(\rho+\alpha_2)]}$]

Proof: See Appendix.

To analyze the dynamic property, following can be shown to hold:

Proposition 6: The open-loop equilibrium $\{g_i^*, r_i^*, m_i^*, b_i^*\}$ is saddle point equilibrium.

Proof: See the Appendix.

5.2 Social Optimum

If the benevolent social planner chooses $g_i(t)$, $g_j(t)$, $r_i(t)$ and $r_j(t)$ to maximize the social welfare, the Hamiltonian function is:

$$\mathcal{H}_i^{so}(t) = e^{-\rho t}\Big[\Big\{\theta\left[\delta\left(m_i(t) + m_j(t)\right) - \frac{\varphi_1}{\tau}\left((g_i(t) + (g_j(t))\right)\right] + (1-\theta)\left[\gamma\left(b_i(t) + b_j(t)\right) - \varphi_2\left((r_i(t) + (r_j(t))\right)\right] + \lambda_i(t)\left[\sqrt{g_i(t) - \alpha_1 g_j(t)} - \alpha_2 m_i(t) + \alpha_3 b_i(t)\right] + \lambda_j(t)\left[\sqrt{g_j(t) - \alpha_1 g_i(t)} - \alpha_2 m_j(t) + \alpha_3 b_j(t)\right] + \psi_i(t)\left[\sqrt{r_i(t) - \beta_1 r_j(t)} - \beta_2 b_i(t)\right] + \psi_j(t)\left[\sqrt{r_j(t) - \beta_1 r_i(t)} - \beta_2 b_j(t)\right]\Big\}\Big] \tag{33}$$

The social optimum exercise is reduced to the following proposition:

Proposition 7: At the social optimum, party i's offer for the expenditure on public good and regulatory benefit are as follows- $g_i^{so} = \frac{(1-\alpha_1)}{4}[\Omega_1\delta]^2$ and $r_i^{so} = \frac{(1-\beta_1)}{4}[\Omega_2\gamma + \Omega_2\Omega_3\delta]^2$. The respective voting support from the citizen voters and the financial contribution by the SIG are - $m_i^{so} = \frac{1}{2\alpha_2}\left[\left\{(1-\beta_1)\Omega_2\frac{\alpha_3}{\beta_2}\right\}\gamma + \left\{(1-\alpha_1)\Omega_1 + (1-\beta_1)\Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\right\}\delta\right]$ and $b_i^{so} = \frac{(1-\beta_1)}{2\beta_2}[\Omega_2\gamma + \Omega_2\Omega_3\delta]$. [Where, $\Omega_1 = \frac{\tau}{\varphi_1(\rho+\alpha_2)}$, $\Omega_2 = \frac{1}{[\varphi_2(\rho+\beta_2)]}$, $\Omega_3 = \frac{\theta\alpha_3}{[(1-\theta)(\rho+\alpha_2)]}$].

Proof: See Appendix.

Proposition 8: The social planner equilibrium $\{g_i^{so}, m_i^{so}, r_i^{so}, b_i^{so}\}$ is a saddle point.

Proof: This is analogous to the proof for the private optimum, hence skipped.

5.3 Private vs Social Optimum

To see the optimum solutions, the difference between the private and social optimum has been observed. The comparative solutions for the private and social optimums are as follows:

$$\mathrm{G}^* = \mathrm{g}_\mathrm{i}^* + \mathrm{g}_\mathrm{j}^* = \frac{1}{2(1-\alpha_1)}[\Omega_1\delta]^2 \tag{34}$$

$$\mathrm{G}^{\mathrm{so}} = \mathrm{g}_\mathrm{i}^{\mathrm{so}} + \mathrm{g}_\mathrm{j}^{\mathrm{so}} = \frac{(1-\alpha_1)}{2}[\Omega_1\delta]^2 = (1-\alpha_1)^2\mathrm{G}^* \tag{35}$$

$$\mathrm{R}^* = \mathrm{r}_\mathrm{i}^* + \mathrm{r}_\mathrm{j}^* = \frac{1}{2(1-\beta_1)}[\Omega_2\gamma + \Omega_2\Omega_3\delta]^2 \tag{36}$$

$$\mathrm{R}^{\mathrm{so}} = \mathrm{r}_\mathrm{i}^{\mathrm{so}} + \mathrm{r}_\mathrm{j}^{\mathrm{so}} = \frac{(1-\beta_1)}{2}[\Omega_2\gamma + \Omega_2\Omega_3\delta]^2 = (1-\beta)^2\mathrm{R}^* \tag{37}$$

$$\mathrm{M}^* = \mathrm{m}_\mathrm{i}^* + \mathrm{m}_\mathrm{j}^* = \frac{1}{\alpha_2}\left[\left(\Omega_2\frac{\alpha_3}{\beta_2}\right)\gamma + \left(\Omega_1 + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\right)\delta\right] \tag{38}$$

$$\mathrm{M}^{\mathrm{so}} = \mathrm{m}_\mathrm{i}^{\mathrm{so}} + \mathrm{m}_\mathrm{j}^{\mathrm{so}} = \frac{1}{\alpha_2}\left[\left\{(1-\beta_1)\Omega_2\frac{\alpha_3}{\beta_2}\right\}\gamma + \left\{(1-\alpha_1)\Omega_1 + (1-\beta_1)\Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\right\}\delta\right] \tag{39}$$

$$\mathrm{B}^* = \mathrm{b}_\mathrm{i}^* + \mathrm{b}_\mathrm{j}^* = \frac{1}{\beta_2}[\Omega_2\gamma + \Omega_2\Omega_3\delta] \tag{40}$$

$$\mathrm{B}^{\mathrm{so}} = \mathrm{b}_\mathrm{i}^{\mathrm{so}} + \mathrm{b}_\mathrm{j}^{\mathrm{so}} = \frac{(1-\beta_1)}{\beta_2}[\Omega_2\gamma + \Omega_2\Omega_3\delta] = (1-\beta_1)\mathrm{B}^* \tag{41}$$

Considering, $0 < \alpha_1, \beta_1 < 1$, clearly we have $\mathrm{G}^* > \mathrm{G}^{\mathrm{so}}$, $\mathrm{R}^* > \mathrm{R}^{\mathrm{so}}$, $\mathrm{M}^* > \mathrm{M}^{\mathrm{so}}$ and $\mathrm{B}^* > \mathrm{B}^{\mathrm{so}}$. Even in the case, when structure of the cost is linear to both, the expenditure on public good $g(t)$ and regulatory benefit $r(t)$, the private optimum is higher than the social optimum. Accordingly, we have $D^* > D^{SO}$. Comparing, Model-I and Model-II, we notice that the private optimum in latter is higher than the former, whereas social optimum is higher in the former than the latter. That means, the higher the offer of expenditure on public good implies, the larger is the vote share to the party and hence the party wins the election. Similarly, higher the regulatory benefit, higher will be the financial contribution by the SIG to political party.

In fact, an attempt by the party to provide larger offer of the expenditure on public good and higher the regulatory benefit to the SIG pushes the deficit of the economy at the higher level. This also confirms the result by Persson (1998).

As the method has been adopted in Model-I, by relaxing the assumptions from both the parties have access to public expenditure, and regulatory power to only the incumbent, does not change the basic conclusion of the Model-II as well. The derivation methods are same as the Model-I and hence are not reported here.

6. Conclusion

In the dynamic analysis of the differential games where there are four players, namely - two political parties (individual), voters and a special interest group. The basic results state that open-loop equilibrium is degenerate closed-loop equilibrium. Therefore, the open-loop equilibrium is a sub-game perfect. The pay-off for all these players has also been solved in a non-cooperative game against the social optimum. We find that, the private optimum is always higher than the social optimum. That is, provision of the public good has a tendency to overspend and incur higher deficits and hence the voters vote retrospectively to the party which overspend and consequently create higher deficits. Similarly, private optimal regulatory benefit helps the political parties to receive higher financial contribution. This further adds to the fiscal deficit due to overspending and can lead to higher gross cost in the economy. Also, by relaxing the assumptions from both the parties' access to public expenditure and regulatory power to only the incumbent, the basic results do not change.

For further research exploration of the topic, we proceed to the 'N' player dynamic games to analyze the private and social optimum. Apart from this, we would like to analyze the optimal number of parties and optimal date of election. In addition to this, we also can further analyse whether the relationships between the corporate interest group and the political party have distributive effect in the economy, particularly in terms of inequality and poverty and will it further lead to plutocracy and oligarchy?

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Appendix

Proof of Proposition 1: In the open-loop formulation, the Hamiltonian for party (politician) 'i' re-write as follows-

$$\mathcal{H}_i(t) = e^{-\rho t}\left\{\theta\left[\delta_i m_i(t) - \frac{\varphi_1}{2}\frac{(g_i(t))^2}{\tau}\right] + (1-\theta)\left[\gamma_i b_i(t) - \frac{\varphi_2}{2}(r_i(t))^2\right] + \lambda_i(t)\left[g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t)\right] + \psi_i(t)\left[r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t)\right]\right\} \quad \text{(a1)}$$

The first order conditions for the open-loop equilibrium are-

$$\frac{\partial \mathcal{H}_i(t)}{\partial g_i(t)} = -\frac{\theta\varphi_1}{\tau} g_i(t) + \lambda_i(t) = 0 \Rightarrow \quad \text{(a2)}$$

$$\frac{\theta\varphi_1}{\tau} g_i(t) = \lambda_i(t) \quad \text{(a3)}$$

$$\frac{\partial \mathcal{H}_i(t)}{\partial r_i(t)} = -(1-\theta)\varphi_2 r_i(t) + \psi_i(t) = 0 \Rightarrow \quad \text{(a4)}$$

$$(1-\theta)\varphi_2 r_i(t) = \psi_i(t) \quad \text{(a5)}$$

$$\dot{\lambda}_i(t) - \rho\lambda_i(t) = -\frac{\partial \mathcal{H}_i(t)}{\partial m_i(t)} \Rightarrow \quad \text{(a6)}$$

$$\dot{\lambda}_i(t) = (\rho + \alpha_2)\lambda_i(t) - \theta\delta_i \quad \text{(a7)}$$

$$\dot{\psi}_i(t) - \rho\psi_i(t) = -\frac{\partial \mathcal{H}_i(t)}{\partial b_i(t)} \Rightarrow \quad \text{(a8)}$$

$$\dot{\psi}_i(t) = (\rho + \beta_2)\psi_i(t) - (1-\theta)\gamma_i - \alpha_3\lambda_i(t) \quad \text{(a9)}$$

The initial conditions are, $m_i(0) = m_{i0}$ and the transversality conditions are $\psi_i(T) = 0$ and $\lambda_i(T) = 0 \forall i$.

Substituting eq. (a3) in eq. (a7) gives,

$$\dot{\lambda}_i(t) = (\rho + \alpha_2)\frac{\theta\varphi_1}{\tau} g_i(t) - \theta\delta_i \quad \text{(a10)}$$

From eq. (a3) and eq. (a10) we can write as,

$$\frac{\partial g_i(t)}{\partial t} \propto \frac{\partial \lambda_i(t)}{\partial t} = (\rho + \alpha_2)\frac{\theta\varphi_1}{\tau} g_i(t) - \theta\delta_i \quad \text{(a11)}$$

For $\dot{\lambda}_i(t) = 0 \Rightarrow \boldsymbol{g_i^*} = \boldsymbol{\Omega_1 \delta_i}$ (a12)

Where, $\Omega_1 = \frac{\tau}{\varphi_1(\rho+\alpha_2)}$.

From eq. (1), the dynamic change in voting support is, $\dot{m}_i(t) = g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t)$ at $\dot{m}_i(t) = 0$ and substituting eq. (a12) in the equation for $\dot{m}_i(t)$ gives,

$$m_i(t) = \frac{\tau}{\alpha_2(\rho+\alpha_2)}\left(\delta_i - \alpha_1\delta_j\right) + \frac{\alpha_3}{\alpha_2} b_i(t) \tag{a13}$$

Substituting eq. (a5) in eq. (a9) gives,

$$\dot{\psi}_i(t) = (\rho + \beta_2)(1-\theta)\varphi_2 r_i(t) - (1-\theta)\gamma_i - \alpha_3\lambda_i(t) \tag{a14}$$

From eq. (a5), eq. (a9) and eq. (a14) we have

$$\frac{\partial r_i(t)}{\partial t} \propto \frac{\partial \psi_i(t)}{\partial t} = (\rho + \beta_2)(1-\theta)\varphi_2 r_i(t) - (1-\theta)\gamma_i - \alpha_3\lambda_i(t) \tag{a15}$$

$$\text{At, } \dot{\psi}_i(t) = 0 \Rightarrow r_i(t) = \frac{\gamma_i}{\varphi_2(\rho+\beta_2)} + \left(\frac{\theta}{1-\theta}\right)\left(\frac{\alpha_3\varphi_1}{\tau\varphi_2(\rho+\beta_2)}\right) g_i(t) \tag{a16}$$

Substituting eq. (a12) in (a16) to get,

$$\boldsymbol{r_i^*} = \frac{1}{\boldsymbol{\varphi_2(\rho+\beta_2)}}\left[\boldsymbol{\gamma_i} + \left(\frac{\boldsymbol{\theta}}{\boldsymbol{1-\theta}}\right)\frac{\boldsymbol{\alpha_3\delta_i}}{\boldsymbol{(\rho+\alpha_2)}}\right] \tag{a17}$$

$$\text{Or, } \boldsymbol{r_i^*} = [\boldsymbol{\Omega_2\gamma_i} + \boldsymbol{\Omega_2\Omega_3\delta_i}] \tag{a18}$$

Where, $\Omega_2 = \frac{1}{[\varphi_2(\rho+\beta_2)]}$, and $\Omega_3 = \frac{\theta\alpha_3}{[(1-\theta)(\rho+\alpha_2)]}$.

From eq. (2), if $\dot{b}_i(t) = 0$ gives,

$$\boldsymbol{b_i^*(t)} = \frac{1}{[\boldsymbol{\beta_2\varphi_2(\rho+\beta_2)}]}\left[\left(\boldsymbol{\gamma_i} - \boldsymbol{\beta_1\gamma_j}\right) + \left(\frac{\boldsymbol{\theta}}{\boldsymbol{1-\theta}}\right)\frac{\boldsymbol{\alpha_3}}{\boldsymbol{(\rho+\alpha_2)}}\left(\boldsymbol{\delta_i} - \boldsymbol{\beta_1\delta_j}\right)\right] \tag{a19}$$

$$\text{Or, } \boldsymbol{b_i^*} = \frac{1}{\boldsymbol{\beta_2}}\left[\boldsymbol{\Omega_2}\left(\boldsymbol{\gamma_i} - \boldsymbol{\beta_1\gamma_j}\right) + \boldsymbol{\Omega_2\Omega_3}\left(\boldsymbol{\delta_i} - \boldsymbol{\beta_1\delta_j}\right)\right] \tag{a20}$$

Now, substituting eq. (a19) in eq. (a13), it gives,

$m_i^*(t) = \frac{\tau}{\alpha_2\varphi_1(\rho+\alpha_2)}(\delta_i - \alpha_1\delta_j) + \frac{\alpha_3}{\alpha_2\beta_2\varphi_2(\rho+\beta_2)}(\gamma_i - \beta_1\gamma_j) + \frac{\theta}{1-\theta}\left[\frac{\alpha_3^2}{\alpha_2\beta_2\varphi_2(\rho+\beta_2)((\rho+\alpha_2)}\right](\delta_i -$
$\beta_1\delta_j)$ (a21)

$$\boldsymbol{m_i^*} = \frac{1}{\boldsymbol{\alpha}_2}\left[\boldsymbol{\Omega}_1(\boldsymbol{\delta_i} - \boldsymbol{\alpha}_1\boldsymbol{\delta_j}) + \boldsymbol{\Omega}_2\frac{\boldsymbol{\alpha}_3}{\boldsymbol{\beta}_2}(\boldsymbol{\gamma_i} - \boldsymbol{\beta}_1\boldsymbol{\gamma_j}) + \boldsymbol{\Omega}_2\boldsymbol{\Omega}_3\frac{\boldsymbol{\alpha}_3}{\boldsymbol{\beta}_2}(\boldsymbol{\delta_i} - \boldsymbol{\beta}_1\boldsymbol{\delta_j})\right] \quad \text{(a22)}$$

Or, $\boldsymbol{m_i^*} = \frac{1}{\boldsymbol{\alpha}_2}\left[\boldsymbol{\Omega}_2\frac{\boldsymbol{\alpha}_3}{\boldsymbol{\beta}_2}(\boldsymbol{\gamma_i} - \boldsymbol{\beta}_1\boldsymbol{\gamma_j}) + \left(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2\boldsymbol{\Omega}_3\frac{\boldsymbol{\alpha}_3}{\boldsymbol{\beta}_2}\right)\boldsymbol{\delta_i} - \left(\boldsymbol{\Omega}_1\boldsymbol{\alpha}_1 + \boldsymbol{\Omega}_2\boldsymbol{\Omega}_3\frac{\boldsymbol{\alpha}_3\boldsymbol{\beta}_1}{\boldsymbol{\beta}_2}\right)\boldsymbol{\delta_j}\right]$ (a23)

Proof of Proposition 2: The required equation of motions are-

$$\dot{m}_i(t) = g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t) \quad \text{(a24)}$$

$$\dot{b}_i(t) = r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t) \quad \text{(a25)}$$

$$\dot{g}_i(t) = \frac{\tau}{\theta\varphi_1}\left[(\rho + \alpha_2)\frac{\theta\varphi_1}{\tau}g_i(t) - \theta\delta_i\right] \quad \text{(a26)}$$

$$\dot{r}_i(t) = \frac{1}{\varphi_2(1-\theta)}[(\rho + \beta_2)(1-\theta)\varphi_2 r_i(t) - (1-\theta)\gamma_i - \alpha_3\lambda_i(t)] \quad \text{(a27)}$$

The stability analysis of the equation from eq. (a24) - eq. (a27) depends on the sign of the trace and determinant of the Jacobian matrix.

$$J = \left\|\begin{bmatrix} -\alpha_2 & \alpha_3 & 1 & 0 \\ 0 & -\beta_2 & 0 & 1 \\ 0 & 0 & \rho + \alpha_2 & 0 \\ 0 & 0 & 0 & \rho + \beta_2 \end{bmatrix}\right\|(g_i^*, r_i^*, m_i^*, b_i^*) \quad \text{(a28)}$$

The trace, $Tr(J) = 2\rho > 0$ and the determinant $\Delta(J) = -\beta_2[(\rho + \alpha_2)(\rho + \beta_2)] < 0$. Since, the determinant of the coefficient matrix of the Jacobian is negative, the characteristic roots must be opposite in sign. Consequently, the optimum solution $(g_i^*, r_i^*, m_i^*, b_i^*)$ has saddle point equilibrium.

Proof of Proposition 3: The Hamiltonian Function for the social optimum is as follows-

$$\mathcal{H}_i^{\mathrm{so}}(t) = e^{-\rho t}\left[\left\{\theta\left[\delta_i m_i(t) + \delta_j m_j(t) - \frac{\varphi_1}{2}\frac{(g_i(t))^2}{\tau} - \frac{\varphi_1}{2}\frac{(g_j(t))^2}{\tau}\right] + (1-\theta)\left[\gamma_i b_i(t) + \gamma_j b_j(t) - \frac{\varphi_2}{2}(r_i(t))^2 - \frac{\varphi_2}{2}(r_j(t))^2\right] + \lambda_i(t)\left[g_i(t) - \alpha_1 g_j(t) - \alpha_2 m_i(t) + \alpha_3 b_i(t)\right] + \lambda_j(t)\left[g_j(t) - \alpha_1 g_i(t) - \alpha_2 m_j(t) + \alpha_3 b_j(t)\right] + \psi_i(t)\left[r_i(t) - \beta_1 r_j(t) - \beta_2 b_i(t)\right] + \psi_j(t)\left[r_j(t) - \beta_1 r_i(t) - \beta_2 b_j(t)\right]\right\}\right] \quad \text{(a29)}$$

The First Order conditions are-

$$\frac{\partial \mathcal{H}_i^{\mathrm{SO}}(t)}{\partial g_i(t)} = -\frac{\theta\varphi_1}{\tau} g_i(t) + \lambda_i(t) - \alpha_1\lambda_j(t) = 0 \Rightarrow \quad \text{(a30)}$$

$$\frac{\theta\varphi_1}{\tau} g_i(t) = \lambda_i(t) - \alpha_1\lambda_j(t) \Rightarrow \quad \text{(a31)}$$

$$\lambda_i = \frac{\theta\varphi_1}{\tau} g_i(t) + \alpha_1\lambda_j(t) \quad \text{(a32)}$$

and, $\dot{\lambda}_i(t) - \rho\lambda_i(t) = -\frac{\partial \mathcal{H}_i^{\mathrm{so}}(t)}{\partial m_i(t)} \Rightarrow$ (a33)

$$\dot{\lambda}_i(t) = (\rho + \alpha_2)\lambda_i(t) - \theta\delta_i \quad \text{(a34)}$$

From eq. (a31) we know that,

$$\frac{\partial g_i(t)}{\partial t} \propto \frac{\partial \lambda_i(t)}{\partial t} - \alpha_1 \frac{\partial \lambda_i(t)}{\partial t} \quad \text{(a35)}$$

From eq. (a32), we also obtain $\lambda_i = \frac{\theta\varphi_1}{\tau} g_i(t) + \alpha_1\lambda_j(t)$ and analogous equation $\lambda_j = \frac{\theta\varphi_1}{\tau} g_j(t) + \alpha_1\lambda_i(t)$. Substituting for we obtain λ_j to get the equation for λ_i is,

$$\lambda_i(t) = \frac{\theta\varphi_1\left[g_i(t) + \alpha_1 g_j(t)\right]}{\tau(1-\alpha_1^2)} \quad \text{(a36)}$$

Using eq. (a36) and (a34), (a35) can re-written as,

$$\frac{\partial g_i(t)}{\partial t} \propto (\rho + \alpha_2)\lambda_i(t) - \theta\delta_i - \alpha_1\left[(\rho + \alpha_2)\lambda_j(t) - \theta\delta_j\right] \quad \text{(a37)}$$

$$\frac{\partial g_i(t)}{\partial t} \propto (\rho + \alpha_2)\left[\lambda_i(t) - \alpha_1\lambda_j\right] - \theta\left[\delta_i - \alpha_1\delta_j\right] \quad \text{(a38)}$$

$$\frac{\partial g_i(t)}{\partial t} \propto \left[(\rho + \alpha_2)\frac{\theta\varphi_1}{\tau}\right] g_i(t) - \theta\left[\delta_i - \alpha_1\delta_j\right] \quad \text{(a39)}$$

If $\frac{\partial g_i(t)}{\partial t} = 0$, we get,

$$\boldsymbol{g_i^{\text{so}}} = \frac{\boldsymbol{\tau}}{\boldsymbol{\varphi_1(\rho+\alpha_2)}}[\boldsymbol{\delta_i} - \boldsymbol{\alpha_1\delta_j}] \tag{a40}$$

Or, $\boldsymbol{g_i^{so}} = \boldsymbol{\Omega_1}[\boldsymbol{\delta_i} - \boldsymbol{\alpha_1\delta_j}]$, (a41)

Where, $\Omega_1 = \frac{\tau}{\varphi_1(\rho+\alpha_2)}$.

The social optimal g_i^{so} from eq. (a40) can be substituted in the eq. (1) to obtain $m_i(t)$. At $\frac{\partial m_i(t)}{\partial t} = 0$, it gives,

$$m_i(t) = \frac{1}{\alpha_2}\left[g_i(t) - \alpha_1 g_j(t) + \alpha_3 b_i(t)\right] \Rightarrow \tag{a42}$$

$$m_i(t) = \frac{1}{\alpha_2}\left[\Omega_1(\delta_i - \alpha_1\delta_j) - \alpha_1\Omega_1(\delta_j - \alpha_1\delta_i) + \alpha_3 b_i(t)\right] \Rightarrow \tag{a43}$$

$$m_i(t) = \frac{1}{\alpha_2}\left[\Omega_1(1 + \alpha_1^2)\delta_i - 2\Omega_1\alpha_1\delta_j + \alpha_3 b_i(t)\right] \tag{a44}$$

Similarly, solution of the regulatory benefit and financial contribution received can be solved as,

$$\frac{\partial \mathcal{H}_i^{\text{so}}(t)}{\partial r_i(t)} = -(1-\theta)\varphi_2 r_i(t) + \psi_i(t) - \beta_1\psi_j(t) = 0 \Rightarrow \tag{a45}$$

$$(1-\theta)\varphi_2 r_i(t) = \psi_i(t) - \beta_1\psi_j(t) \Rightarrow \tag{a46}$$

$$\psi_i(t) = (1-\theta)\varphi_2 r_i(t) + \beta_1\psi_j(t) \tag{a47}$$

and, $\dot{\psi}_i(t) - \rho\psi_i(t) = -\frac{\partial \mathcal{H}_i^{so}(t)}{\partial b_i(t)} \Rightarrow$ (a48)

$$\dot{\psi}_i(t) = (\rho + \beta_2)\psi_i(t) - (1-\theta)\gamma_i - \alpha_3\lambda_i(t) \tag{a49}$$

Equation (a29) can be re-written as,

$$\frac{\partial r_i(t)}{\partial t} \propto \frac{\partial \psi_i(t)}{\partial t} - \beta_1\frac{\partial \psi_j(t)}{\partial t} \tag{a50}$$

From eq. (a47), we also obtain $\psi_i(t) = (1-\theta)\varphi_2 r_i(t) + \beta_1\psi_j(t)$ and analogously we can write $\psi_j(t) = (1-\theta)\varphi_2 r_j(t) + \beta_1\psi_i(t)$. Substitute $\psi_j(t)$ to get an equation for $\psi_i(t)$ as,

$$\psi_i = \frac{(1-\theta)\varphi_2}{(1-\beta_1^2)}\left[r_i(t) + \alpha_1 r_j(t)\right] \tag{a51}$$

Using eq. (a51) and eq. (a49), eq. (a50) can re-written as,

$$\frac{\partial r_i(t)}{\partial t} \propto (\rho+\beta_2)\psi_i(t) - (1-\theta)\gamma_i - \alpha_3\lambda_i(t) - \beta_1\left[(\rho+\beta_2)\psi_j(t) - (1-\theta)\gamma_j - \alpha_3\lambda_j(t)\right] \tag{a52}$$

$$\frac{\partial r_i(t)}{\partial t} \propto (\rho+\beta_2)\left[\psi_i(t) - \beta_1\psi_j(t)\right] - (1-\theta)\left[\gamma_i - \beta_1\gamma_j\right] - \alpha_3\left[\lambda_i(t) - \beta_1\lambda_j(t)\right] \tag{a53}$$

$$\frac{\partial r_i(t)}{\partial t} \propto [(\rho+\beta_2)(1-\theta)\varphi_2]r_i(t) - (1-\theta)\left[\gamma_i - \beta_1\gamma_j\right] - \alpha_3\left[\lambda_i(t) - \beta_1\lambda_j(t)\right] \tag{a54}$$

If $\frac{\partial r_i(t)}{\partial t} = 0$, we get, $r_i(t) = \frac{(1-\theta)[\gamma_i - \beta_1\gamma_j] + \alpha_3[\lambda_i(t) - \beta_1\lambda_j(t)]}{[\varphi_2(\rho+\beta_2)(1-\theta)]}$ (a55)

From equation (a34), $\lambda_i(t) = \frac{\theta\delta_i}{(\rho+\alpha_2)}$ at $\dot{\lambda}_i(t) = 0$. Substituting this in eq. (a55) we get,

$$r_i(t) = \frac{(1-\theta)[\gamma_i - \beta_1\gamma_j] + \alpha_3\left[\frac{\theta\delta_i}{(\rho+\alpha_2)} - \beta_1\frac{\theta\delta_j}{(\rho+\alpha_2)}\right]}{[\varphi_2(\rho+\beta_2)(1-\theta)]} \tag{a56}$$

$$r_i(t) = \frac{1}{[\varphi_2(\rho+\beta_2)]}\left[\gamma_i - \beta_1\gamma_j\right] + \frac{\theta\alpha_3}{[1-\theta)\varphi_2(\rho+\alpha_2)(\rho+\beta_2)]}\left[\delta_i - \beta_1\delta_j\right] \tag{a57}$$

$$\boldsymbol{r_i^{so}(t)} = \frac{1}{[\boldsymbol{\varphi_2(\rho+\beta_2)}]}\left[\left(\boldsymbol{\gamma_i} - \boldsymbol{\beta_1\gamma_j}\right) + \frac{\boldsymbol{\theta\alpha_3}}{[\boldsymbol{1-\theta)(\rho+\alpha_2)}]}\left(\boldsymbol{\delta_i} - \boldsymbol{\beta_1\delta_j}\right)\right] \tag{a58}$$

Or, $\boldsymbol{r_i^{so}(t)} = \left[\boldsymbol{\Omega_2}\left(\boldsymbol{\gamma_i} - \boldsymbol{\beta_1\gamma_j}\right) + \boldsymbol{\Omega_2\Omega_3}\left(\boldsymbol{\delta_i} - \boldsymbol{\beta_1\delta_j}\right)\right]$ (a59)

Where, $\Omega_2 = \frac{1}{[\varphi_2(\rho+\beta_2)]}$ and $\Omega_3 = \frac{\theta\alpha_3}{[1-\theta)(\rho+\alpha_2)]}$

$r_i^{so}(t)$ can be substituted in the dynamic equation of $b_i(t)$ of eq. (2). If $\dot{b}_i(t) = 0$

$$\Rightarrow b_i(t) = \frac{[r_i(t) - \beta_1 r_j(t)]}{\beta_2} \tag{a60}$$

$$b_i(t) = \frac{1}{\beta_2}\left[\Omega_2\left(\gamma_i - \beta_1\gamma_j\right) + \Omega_2\Omega_3\left(\delta_i - \beta_1\delta_j\right) - \beta_1\left[\Omega_2\left(\gamma_j - \beta_1\gamma_i\right) + \Omega_2\Omega_3\left(\delta_j - \beta_1\delta_i\right)\right]\right] \tag{a61}$$

$$\boldsymbol{b_i^{so}} = \frac{1}{\boldsymbol{\beta_2}}\left[\boldsymbol{\Omega_2}(1+\boldsymbol{\beta_1^2})(\boldsymbol{\gamma_i} + \boldsymbol{\Omega_3\delta_i}) - 2\boldsymbol{\Omega_2\beta_1}(\boldsymbol{\gamma_j} + \boldsymbol{\Omega_3\delta_j})\right] \tag{a62}$$

Or, $\boldsymbol{b_i^{so}} = \frac{1}{\boldsymbol{\beta_2}}\left[\boldsymbol{\Omega_2}\left((1+\boldsymbol{\beta_1^2})\boldsymbol{\gamma_i} - 2\boldsymbol{\beta_1\gamma_j}\right) + \boldsymbol{\Omega_2\Omega_3}\left((1+\boldsymbol{\beta_1^2})\boldsymbol{\delta_i} - 2\boldsymbol{\beta_1\delta_j}\right)\right]$ (a63)

Substituting eq. (a63) in eq. (a44) gives,

$$m_i(t) = \frac{1}{\alpha_2}\left[\Omega_1\left[(1+\alpha_1^2)\delta_i - 2\alpha_1\delta_j\right] + \frac{\alpha_3}{\beta_2}\left[\Omega_2\left((1+\beta_1^2)\gamma_i - 2\beta_1\gamma_j\right) + \Omega_2\Omega_3\left((1+\beta_1^2)\delta_i - 2\beta_1\delta_j\right)\right]\right] \quad \text{(a64)}$$

$$\boldsymbol{m_i^{so}} = \frac{1}{\boldsymbol{\alpha_2}}\left[\boldsymbol{\Omega_2}\frac{\boldsymbol{\alpha_3}}{\boldsymbol{\beta_2}}\left((1+\boldsymbol{\beta_1^2})\boldsymbol{\gamma_i} - 2\boldsymbol{\beta_1}\boldsymbol{\gamma_j}\right) + \left[\boldsymbol{\Omega_1}(1+\boldsymbol{\alpha_1^2}) + \boldsymbol{\Omega_2}\boldsymbol{\Omega_3}\frac{\boldsymbol{\alpha_3}}{\boldsymbol{\beta_2}}(1+\boldsymbol{\beta_1^2})\right]\boldsymbol{\delta_i} - 2\left[\boldsymbol{\Omega_1}\boldsymbol{\alpha_1} + \boldsymbol{\Omega_2}\boldsymbol{\Omega_3}\frac{\boldsymbol{\alpha_3}\boldsymbol{\beta_1}}{\boldsymbol{\beta_2}}\right]\boldsymbol{\delta_j}\right] \quad \text{(a65)}$$

$$\Rightarrow \boldsymbol{m_i^{so}} = \frac{1}{\boldsymbol{\alpha_2}}\left[\boldsymbol{\Omega_2}\frac{\boldsymbol{\alpha_3}}{\boldsymbol{\beta_2}}\left((1+\boldsymbol{\beta_1^2})\boldsymbol{\gamma_i} - 2\boldsymbol{\beta_1}\boldsymbol{\gamma_j}\right) + \boldsymbol{\Omega_4}\boldsymbol{\delta_i} - 2\boldsymbol{\Omega_5}\boldsymbol{\delta_j}\right] \quad \text{(a66)}$$

Where, $\Omega_4 = \left[\Omega_1(1+\alpha_1^2) + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}(1+\beta_1^2)\right]$ and $\Omega_5 = \left[\Omega_1\alpha_1 + \Omega_2\Omega_3\frac{\alpha_3\beta_1}{\beta_2}\right]$

Proof of Proposition 5: In the open-loop formulation, the Hamiltonian for party (politician) 'i' re-write as follows-

$$\mathcal{H}_i(t) = e^{-\rho t}\left\{\theta\left[\delta m_i(t) - \varphi_1\frac{g_i(t)}{\tau}\right] + (1-\theta)[\gamma b_i(t) - \varphi_2 r_i(t)] + \lambda_i(t)\left[\sqrt{g_i(t) - \alpha_1 g_j(t)} - \alpha_2 m_i(t) + \alpha_3 b_i(t)\right] + \psi_i(t)\left[\sqrt{r_i(t) - \beta_1 r_j(t)} - \beta_2 b_i(t)\right]\right\} \quad \text{(a67)}$$

The first order conditions for the open-loop equilibrium are-

$$\frac{\partial\mathcal{H}_i(t)}{\partial g_i(t)} = -\frac{\theta\varphi_1}{\tau} + \frac{\lambda_i(t)}{2\sqrt{g_i(t)-\alpha_1 g_j(t)}} = 0 \Rightarrow \quad \text{(a68)}$$

$$\lambda_i(t) = \frac{2\theta\varphi_1}{\tau}\sqrt{g_i(t) - \alpha_1 g_j(t)} \quad \text{(a69)}$$

$$\dot{\lambda}_i(t) - \rho\lambda_i(t) = -\frac{\partial\mathcal{H}_i(t)}{\partial m_i(t)} \Rightarrow \quad \text{(a70)}$$

$$\dot{\lambda}_i(t) - (\rho + \alpha_2)\lambda_i(t) = \theta\delta \quad \text{(a71)}$$

Plus the initial conditions $m_i(0) = m_{i0}$ and the TVC, $\lambda_i(T) = 0 \forall i$. Similarly, we also have

$$\frac{\partial\mathcal{H}_i(t)}{\partial r_i(t)} = -(1-\theta)\varphi_2 + \frac{\psi_i(t)}{2\sqrt{r_i(t)-\beta_1 r_j(t)}} = 0 \Rightarrow \quad \text{(a72)}$$

$$\psi_i(t) = 2(1-\theta)\varphi_2\sqrt{r_i(t) - \beta_1 r_j(t)} \tag{a73}$$

$$\dot{\psi}_i(t) - \rho\psi_i(t) = -\frac{\partial \mathcal{H}_i(t)}{\partial b_i(t)} \Rightarrow \tag{a74}$$

$$\dot{\psi}_i(t) = (\rho + \beta_2)\psi_i(t) - (1-\theta)\gamma - \alpha_3\lambda_i(t) \tag{a75}$$

And the initial conditions are, $b_i(0) = b_{i0}$ and the Transversality conditions are $\psi_i(T) = 0 \forall i$.

From equation (a68),

$$\lambda_i(t) = \frac{2\theta\varphi_1}{\tau}\sqrt{g_i(t) - \alpha_1 g_j(t)} \Rightarrow \tag{a76}$$

$$g_i(t) = \alpha_1 g_j(t) + \left[\frac{\tau\lambda_i(t)}{2\theta\varphi_1}\right]^2 \tag{a77}$$

Differentiating eq. (a77) w.r.t time to yield,

$$\frac{\partial g_i(t)}{\partial t} = \alpha_1 \frac{\partial g_j(t)}{\partial t} + 2\lambda_i\left[\frac{\tau}{2\theta\varphi_1}\right]^2 \frac{\partial \lambda_i(t)}{\partial t} \tag{a78}$$

By imposing the condition of symmetry, we can have $g_i(t) = g_j(t)$ accordingly, $\frac{\partial g_i(t)}{\partial t} = \frac{\partial g_j(t)}{\partial t}$.

Using eq. (a69) and eq. (a71), eq. (a78) can be re-written as,

$$\frac{\partial g_i(t)}{\partial t} \propto \frac{\partial \lambda_i(t)}{\partial t}$$

$$\propto \frac{2\theta\varphi_1(\rho+\alpha_2)}{\tau}\sqrt{g_i(t) - \alpha_1 g_j(t)} - \theta\delta \tag{a79}$$

$$\frac{\partial g_i(t)}{\partial t} \propto \frac{2\theta\varphi_1(\rho+\alpha_2)}{\tau}\sqrt{(1-\alpha_1)g_i(t)} - \theta\delta \tag{a80}$$

$$\text{If, } \frac{\partial g_i(t)}{\partial t} = 0 \Rightarrow \frac{2\theta\varphi_1(\rho+\alpha_2)}{\tau}\sqrt{(1-\alpha_1)g_i(t)} = \theta\delta \tag{a81}$$

$$\Rightarrow \boldsymbol{g}^* = \frac{1}{(1-\boldsymbol{\alpha}_1)}\left[\frac{\boldsymbol{\tau\delta}}{2\boldsymbol{\varphi}_1(\boldsymbol{\rho}+\boldsymbol{\alpha}_2)}\right]^2 \tag{a82}$$

$$\text{Or, } \boldsymbol{g}^* = \frac{1}{4(1-\boldsymbol{\alpha}_1)}[\boldsymbol{\Omega}_1\boldsymbol{\delta}]^2 \tag{a83}$$

where, $\Omega_1 = \frac{\tau}{\varphi_1(\rho+\alpha_2)}$

Substituting eq. (a83) in to eq. (26) for $m_i(t)$. At $m_i(t) = 0$ gives,

$$\Rightarrow m_i(t) = \frac{1}{\alpha_2}\left[\sqrt{(1-\alpha_1)g_i(t)} + \alpha_3 b_i(t)\right] \quad \text{(a84)}$$

$$\Rightarrow m_i(t) = \frac{1}{\alpha_2}\left[\frac{\tau\delta}{2\varphi_1(\rho+\alpha_2)} + \alpha_3 b_i(t)\right] \quad \text{(a85)}$$

Similarly, from eq. (a73),

$$\psi_i(t) = 2(1-\theta)\varphi_2\sqrt{r_i(t) - \beta_1 r_j(t)} \Rightarrow \quad \text{(a86)}$$

$$r_i(t) = \beta_1 r_j(t) + \left[\frac{\psi_i(t)}{2(1-\theta)\varphi_2}\right]^2 \quad \text{(a87)}$$

Differentiating eq. (a87) w.r.t. time gives,

$$\frac{\partial r_i(t)}{\partial t} = \beta_1\frac{\partial r_j(t)}{\partial t} + 2\psi_i(t)\left[\frac{1}{2(1-\theta)\varphi_2}\right]^2\frac{\partial\psi_i(t)}{\partial t} \quad \text{(a88)}$$

By imposing the condition of symmetry, $r_i(t) = r_j(t)$ and accordingly, $\frac{\partial r_i(t)}{\partial t} = \frac{\partial r_j(t)}{\partial t}$

Using eq. (a73) and eq. (a75), eq. (a88) can be re-written as,

$$\frac{\partial r_i(t)}{\partial t} \propto \frac{\partial\psi_i(t)}{\partial t}$$

$$\propto (\rho+\beta_2)\psi_i(t) - (1-\theta)\gamma - \alpha_3\lambda_i(t) \Rightarrow \quad \text{(a89)}$$

$$\frac{\partial r_i(t)}{\partial t} \propto 2(\rho+\beta_2)(1-\theta)\varphi_2\sqrt{r_i(t) - \beta_1 r_j(t)} - (1-\theta)\gamma - \alpha_3\frac{2\theta\varphi_1}{\tau}\sqrt{g_i(t) - \alpha_1 g_j(t)} \quad \text{(a90)}$$

$$\frac{\partial r_i(t)}{\partial t} \propto 2(\rho+\beta_2)(1-\theta)\varphi_2\sqrt{(1-\beta_1)r_i(t)} - \alpha_3\frac{2\theta\varphi_1}{\tau}\sqrt{(1-\alpha_1)g_i(t)} - (1-\theta)\gamma \quad \text{(a91)}$$

If $\frac{\partial r_i(t)}{\partial t} = 0$, gives,

$$2(\rho+\beta_2)(1-\theta)\varphi_2\sqrt{(1-\beta_1)r_i(t)} = \alpha_3\frac{2\theta\varphi_1}{\tau}\sqrt{(1-\alpha_1)g_i(t)} + (1-\theta)\gamma \quad \text{(a92)}$$

$$\Rightarrow 2\varphi_2(1-\theta)(\rho+\beta_2)\sqrt{(1-\beta_1)r_i(t)} = \frac{\theta\alpha_3\delta}{(\rho+\alpha_2)} + (1-\theta)\gamma \quad \text{(a93)}$$

$$\Rightarrow \sqrt{(1-\beta_1)r_i(t)} = \frac{1}{2\varphi_2(1-\theta)(\rho+\beta_2)}\left[\frac{\theta\alpha_3\delta}{(\rho+\alpha_2)} + (1-\theta)\gamma\right] \tag{a94}$$

$$\Rightarrow \boldsymbol{r_i}^* = \frac{1}{4(1-\boldsymbol{\beta}_1)}\left[\frac{\boldsymbol{\gamma}}{\boldsymbol{\varphi}_2(\boldsymbol{\rho}+\boldsymbol{\beta}_2)} + \frac{\boldsymbol{\theta\alpha}_3\boldsymbol{\delta}}{\boldsymbol{\varphi}_2(1-\boldsymbol{\theta})(\boldsymbol{\rho}+\boldsymbol{\beta}_2)(\boldsymbol{\rho}+\boldsymbol{\alpha}_2)}\right]^2 \tag{a95}$$

$$\text{Or, } \boldsymbol{r_i}^* = \frac{1}{4(1-\boldsymbol{\beta}_1)}[\boldsymbol{\Omega}_2\boldsymbol{\gamma} + \boldsymbol{\Omega}_2\boldsymbol{\Omega}_3\boldsymbol{\delta}]^2 \tag{a96}$$

where, $\Omega_2 = \frac{\gamma}{\varphi_2(\rho+\beta_2)}$ and $\Omega_3 = \frac{\theta\alpha_3}{(1-\theta)(\rho+\alpha_2)}$.

Substituting eq. (a95) in eq. (27). At, $\dot{b}_i(t) = 0$ gives,

$$\Rightarrow \sqrt{(1-\beta_1)r_i(t)} = \beta_2 b_i(t) \tag{a97}$$

$$\boldsymbol{b_i}^* = \frac{1}{2\boldsymbol{\beta}_2}\left[\frac{\boldsymbol{\gamma}}{\boldsymbol{\varphi}_2(\boldsymbol{\rho}+\boldsymbol{\beta}_2)} + \frac{\boldsymbol{\theta\alpha}_3\boldsymbol{\delta}}{\boldsymbol{\varphi}_2(1-\boldsymbol{\theta})(\boldsymbol{\rho}+\boldsymbol{\beta}_2)(\boldsymbol{\rho}+\boldsymbol{\alpha}_2)}\right] \tag{a98}$$

$$\text{Or, } \boldsymbol{b_i}^* = \frac{1}{2\boldsymbol{\beta}_2}[\boldsymbol{\Omega}_2\boldsymbol{\gamma} + \boldsymbol{\Omega}_2\boldsymbol{\Omega}_3\boldsymbol{\delta}] \tag{a99}$$

Substituting eq. (a98) in eq. (a85),

$$m_i(t) = \frac{1}{\alpha_2}\left[\frac{\tau\delta}{2\varphi_1(\rho+\alpha_2)} + \frac{\alpha_3}{2\beta_2}\left\{\frac{\gamma}{\varphi_2(\rho+\beta_2)} + \frac{\theta\alpha_3\delta}{\varphi_2(1-\theta)(\rho+\beta_2)(\rho+\alpha_2)}\right\}\right] \tag{a100}$$

$$\Rightarrow m_i^* = \frac{1}{2\alpha_2}\left[\frac{\alpha_3}{\beta_2\varphi_2(\rho+\beta_2)}\right]\gamma + \frac{1}{2\alpha_2}\left[\frac{\tau}{\varphi_1(\rho+\alpha_2)} + \frac{\theta\alpha_3^2}{\beta_2\varphi_2(1-\theta)(\rho+\beta_2)(\rho+\alpha_2)}\right]\delta \tag{a101}$$

$$\Rightarrow m_i^* = \frac{1}{2\alpha_2}\left[\Omega_2\frac{\alpha_3}{\beta_2}\right]\gamma + \frac{1}{2\alpha_2}\left[\Omega_1 + \Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\right]\delta \tag{a102}$$

$$\Rightarrow \boldsymbol{m_i^*} = \frac{1}{2\boldsymbol{\alpha}_2}\left[\left(\boldsymbol{\Omega}_2\frac{\boldsymbol{\alpha}_3}{\boldsymbol{\beta}_2}\right)\boldsymbol{\gamma} + \left(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2\boldsymbol{\Omega}_3\frac{\boldsymbol{\alpha}_3}{\boldsymbol{\beta}_2}\right)\boldsymbol{\delta}\right] \tag{a103}$$

This proves the proposition

Proof of Proposition 6: The required equation of motions are-

$$\dot{m}_i(t) = \sqrt{(1-\alpha_1)g_i(t)} - \alpha_2 m_i(t) + \alpha_3 b_i(t) \tag{a104}$$

$$\dot{b}_i(t) = \sqrt{(1-\beta_1)r_i(t)} - \beta_2 b_i(t) \tag{a105}$$

$$\dot{g}_i(t) = \frac{2\theta\varphi_1(\rho+\alpha_2)}{\tau}\sqrt{(1-\alpha_1)g_i(t)} - \theta\delta \tag{a106}$$

$$\dot{r}_i(t) = 2(\rho+\beta_2)(1-\theta)\varphi_2\sqrt{(1-\beta_1)r_i(t)} - \alpha_3\frac{2\theta\varphi_1}{\tau}\sqrt{(1-\alpha_1)g_i(t)} - (1-\theta)\gamma \tag{a107}$$

The stability analysis of the equation from eq. (a104) - eq. (a107) depends on the sign of the trace and determinant of the Jacobian matrix.

$$J = \begin{bmatrix} -\alpha_2 & \alpha_3 & \frac{1}{2\sqrt{(1-\alpha_1)g_i(t)}} & 0 \\ 0 & -\beta_2 & 0 & \frac{1}{2\sqrt{(1-\beta_1)r_i(t)}} \\ 0 & 0 & \frac{\theta\varphi_1(\rho+\alpha_2)}{\tau\sqrt{(1-\alpha_1)g_i(t)}} & 0 \\ 0 & 0 & \frac{\theta\varphi_1\alpha_3}{\tau\sqrt{(1-\alpha_1)g_i(t)}} & \frac{\varphi_2(1-\theta)(\rho+\beta_2)}{\sqrt{(1-\beta_1)r_i(t)}} \end{bmatrix}\Bigg|_{(g_i^*, r_i^*, m_i^*, b_i^*)} \tag{a108}$$

The trace, $Tr(J) = -\alpha_2 - \beta_2 + \frac{\theta\varphi_1(\rho+\alpha_2)}{\tau\sqrt{(1-\alpha_1)g_i(t)}} + \frac{\varphi_2(1-\theta)(\rho+\beta_2)}{\sqrt{(1-\beta_1)r_i(t)}} > 0$ and the determinant $\Delta(J) = -\beta_2\left[\frac{\theta\varphi_1(\rho+\alpha_2)}{\tau\sqrt{(1-\alpha_1)g_i(t)}}.\frac{\varphi_2(1-\theta)(\rho+\beta_2)}{\sqrt{(1-\beta_1)r_i(t)}}\right] < 0$. Since, the determinant of the coefficient matrix of the Jacobian is negative, the characteristic roots must be opposite in sign. Consequently, the optimum solution, $(g_i^*, r_i^*, m_i^*, b_i^*)$ has saddle point equilibrium.

Proof of Proposition 7: The Hamiltonian function for the social optimum can re-write as follows-

$$\mathcal{H}_i^{so}(t) = e^{-\rho t}\Big[\Big\{\theta\left[\delta\left(m_i(t)+m_j(t)\right) - \frac{\varphi_1}{\tau}\left((g_i(t) + (g_j(t))\right)\right] + (1-\theta)\left[\gamma\left(b_i(t)+b_j(t)\right) - \varphi_2\left((r_i(t) + (r_j(t))\right)\right] + \lambda_i(t)\left[\sqrt{g_i(t) - \alpha_1 g_j(t)} - \alpha_2 m_i(t) + \alpha_3 b_i(t)\right] + \lambda_j(t)\left[\sqrt{g_j(t) - \alpha_1 g_i(t)} - \alpha_2 m_j(t) + \alpha_3 b_j(t)\right] + \psi_i(t)\left[\sqrt{r_i(t) - \beta_1 r_j(t)} - \beta_2 b_i(t)\right] + \psi_j(t)\left[\sqrt{r_j(t) - \beta_1 r_i(t)} - \beta_2 b_j(t)\right]\Big\}\Big] \tag{a109}$$

The first order conditions for the open-loop equilibrium are-

$$\frac{\partial\mathcal{H}_i^{so}(t)}{\partial g_i(t)} = -\frac{\theta\varphi_1}{\tau} + \frac{\lambda_i(t)}{2\sqrt{g_i(t)-\alpha_1 g_j(t)}} - \frac{\alpha_1\lambda_j(t)}{2\sqrt{g_j(t)-\alpha_1 g_i(t)}} = 0 \Rightarrow \tag{a110}$$

$$\dot{\lambda}_i(t) - \rho\lambda_i(t) = -\frac{\partial\mathcal{H}_i^{so}(t)}{\partial m_i(t)} \Rightarrow \tag{a111}$$

$$\dot{\lambda}_i(t) - \rho\lambda_i(t) = -\theta\delta - \alpha_2\lambda_i(t) \quad \text{(a112)}$$

$$\dot{\lambda}_i(t) - (\rho + \alpha_2)\lambda_i(t) = -\theta\delta \quad \text{(a113)}$$

Similarly,

$$\frac{\partial \mathcal{H}_i^{so}(t)}{\partial r_i(t)} = -(1-\theta)\varphi_2 + \frac{\psi_i(t)}{2\sqrt{r_i(t)-\beta_1 r_j(t)}} - \frac{\beta_1\psi_j(t)}{2\sqrt{r_j(t)-\beta_1 r_i(t)}} = 0 \Rightarrow \quad \text{(a114)}$$

$$\dot{\psi}_i(t) - \rho\psi_i(t) = -\frac{\partial \mathcal{H}_i^{so}(t)}{\partial b_i(t)} \Rightarrow \quad \text{(a115)}$$

$$\dot{\psi}_i(t) - \rho\psi_i(t) = -(1-\theta)\gamma - \alpha_3\lambda_i(t) + \beta_2\psi_i(t) \quad \text{(a116)}$$

$$\dot{\psi}_i(t) - (\rho + \beta_2)\psi_i(t) = -(1-\theta)\gamma - \alpha_3\lambda_i(t) \quad \text{(a117)}$$

Using symmetry condition and from the eq. (a110)

$$-\frac{\theta\varphi_1}{\tau} + \frac{(1-\alpha_1)\lambda_i(t)}{2\sqrt{(1-\alpha_1)g_i(t)}} = 0 \Rightarrow \quad \text{(a118)}$$

$$\lambda_i(t) = \frac{2\theta\varphi_1}{\tau(1-\alpha_1)}\sqrt{(1-\alpha_1)g_i(t)} \quad \text{(a119)}$$

$$\Rightarrow g_i(t) = \frac{1}{(1-\alpha_1)}\left[\frac{\tau(1-\alpha_1)\lambda_i(t)}{2\theta\varphi_1}\right]^2 \quad \text{(a120)}$$

$$\Rightarrow g_i(t) = (1-\alpha_1)\left[\frac{\tau\lambda_i(t)}{2\theta\varphi_1}\right]^2 \quad \text{(a121)}$$

From eq. (a121) and eq. (a113) we get following,

$$\frac{\partial g_i(t)}{\partial t} \propto \frac{\partial \lambda_i(t)}{\partial t} \quad \text{(a122)}$$

$$\propto (\rho + \alpha_2)\lambda_i(t) - \theta\delta \quad \text{(a123)}$$

$$\frac{\partial g_i(t)}{\partial t} \propto (\rho + \alpha_2)\frac{2\theta\varphi_1}{\tau(1-\alpha_1)}\sqrt{(1-\alpha_1)g_i(t)} - \theta\delta \quad \text{(a124)}$$

At, $\frac{\partial g_i(t)}{\partial t} = 0$,

$$\Rightarrow \sqrt{(1-\alpha_1)g_i(t)} = \frac{\delta\tau(1-\alpha_1)}{2\varphi_1(\rho+\alpha_2)} \quad \text{(a125)}$$

$$\Rightarrow \boldsymbol{g_i^{\mathrm{SO}}} = (1-\boldsymbol{\alpha}_1)\left[\frac{\boldsymbol{\delta\tau}}{2\boldsymbol{\varphi}_1(\boldsymbol{\rho}+\boldsymbol{\alpha}_2)}\right]^2 \tag{a126}$$

Or, $\boldsymbol{g_i^{\mathrm{SO}}} = \frac{(1-\boldsymbol{\alpha}_1)}{4}[\boldsymbol{\Omega}_1\boldsymbol{\delta}]^2$ (a127)

Using symmetry condition and eq. (a126) in eq. (26) gives,

$$\dot{m}_i(t) = \frac{1}{\alpha_2}\left[\sqrt{(1-\alpha_1)g_i(t)} + \alpha_3 b_i(t)\right] \Rightarrow \tag{a128}$$

$$\dot{m}_i(t) = \frac{(1-\alpha_1)\delta\tau}{2\varphi_1(\rho+\alpha_2)} - \alpha_2 m_i(t) + \alpha_3 b_i(t) \tag{a129}$$

At, $\dot{m}_i(t) = 0 \Rightarrow m_i(t) = \frac{(1-\alpha_1)\delta\tau}{2\varphi_1\alpha_2(\rho+\alpha_2)} + \frac{\alpha_3}{\alpha_2}b_i(t)$ (a130)

Using symmetry condition and from the eq. (a114)

$$\Rightarrow -(1-\theta)\varphi_2 + \frac{\psi_i(t)}{2\sqrt{r_i(t)-\beta_1 r_j(t)}} - \frac{\beta_1\psi_j(t)}{2\sqrt{r_j(t)-\beta_1 r_i(t)}} = 0 \tag{a131}$$

$$\Rightarrow (1-\theta)\varphi_2 = \frac{(1-\beta_1)\psi_i(t)}{2\sqrt{(1-\beta_1)r_i(t)}} \tag{a132}$$

$$\Rightarrow \psi_i(t) = \frac{2(1-\theta)\varphi_2}{(1-\beta_1)}\sqrt{(1-\beta_1)r_i(t)} \tag{a133}$$

$$\Rightarrow r_i(t) = (1-\beta_1)\left[\frac{\psi_i(t)}{2(1-\theta)\varphi_2}\right]^2 \tag{a134}$$

From eq. (a117) and eq. (a119), eq. (a134) can be written as,

$$\frac{\partial r_i(t)}{\partial t} \propto \frac{\partial \psi_i(t)}{\partial t} \tag{a135}$$

$$\frac{\partial r_i(t)}{\partial t} \propto (\rho+\beta_2)\psi_i(t) - (1-\theta)\gamma - \alpha_3\lambda_i(t) \tag{a136}$$

$$\frac{\partial r_i(t)}{\partial t} \propto (\rho+\beta_2)\frac{2(1-\theta)\varphi_2}{(1-\beta_1)}\sqrt{(1-\beta_1)r_i(t)} - (1-\theta)\gamma - \alpha_3\frac{2\theta\varphi_1}{\tau(1-\alpha_1)}\sqrt{(1-\alpha_1)g_i(t)} \tag{a137}$$

At $\frac{\partial r_i(t)}{\partial t} = 0$,

$$\Rightarrow (\rho+\beta_2)\frac{2(1-\theta)\varphi_2}{(1-\beta_1)}\sqrt{(1-\beta_1)r_i(t)} = (1-\theta)\gamma + \frac{\theta\alpha_3\delta}{(\rho+\alpha_2)} \tag{a138}$$

$$\Rightarrow \sqrt{(1-\beta_1)r_i(t)} = \frac{(1-\beta_1)}{2\varphi_2(1-\theta)(\rho+\beta_2)}\left[(1-\theta)\gamma + \frac{\theta\alpha_3\delta}{(\rho+\alpha_2)}\right] \quad (a139)$$

$$\Rightarrow \sqrt{(1-\beta_1)r_i(t)} = \frac{(1-\beta_1)}{2}\left[\frac{1}{\varphi_2(\rho+\beta_2)}\gamma + \frac{\theta\alpha_3}{\varphi_2(1-\theta)(\rho+\beta_2)(\rho+\alpha_2)}\delta\right] \quad (a140)$$

$$\boldsymbol{r_i^{so}} = \frac{(1-\boldsymbol{\beta}_1)}{4}\left[\frac{1}{\boldsymbol{\varphi}_2(\boldsymbol{\rho}+\boldsymbol{\beta}_2)}\boldsymbol{\gamma} + \frac{\boldsymbol{\theta\alpha}_3}{\boldsymbol{\varphi}_2(1-\boldsymbol{\theta})(\boldsymbol{\rho}+\boldsymbol{\beta}_2)(\boldsymbol{\rho}+\boldsymbol{\alpha}_2)}\boldsymbol{\delta}\right]^2 (a141)$$

$$\text{Or ,}\boldsymbol{r_i^{so}} = \frac{(1-\boldsymbol{\beta}_1)}{4}[\boldsymbol{\Omega}_2\boldsymbol{\gamma} + \boldsymbol{\Omega}_2\boldsymbol{\Omega}_3\boldsymbol{\delta}]^2 \quad (a142)$$

Substituting eq. (a141 in eq. (27) gives,

$$\text{At, } b_i(t) = 0 \Rightarrow \beta_2 b_i(t) = \sqrt{(1-\beta_1)r_i(t)} \quad (a143)$$

$$\Rightarrow \beta_2 b_i(t) = \frac{(1-\beta_1)}{2\varphi_2(1-\theta)(\rho+\beta_2)}\left[(1-\theta)\gamma + \frac{\theta\alpha_3}{(\rho+\alpha_2)}\right] \quad (a144)$$

$$\boldsymbol{b_i}^{\boldsymbol{so}} = \frac{(1-\boldsymbol{\beta}_1)}{2\boldsymbol{\beta}_2}\left[\frac{\boldsymbol{\gamma}}{\boldsymbol{\varphi}_2(\boldsymbol{\rho}+\boldsymbol{\beta}_2)} + \frac{\boldsymbol{\theta\alpha}_3\boldsymbol{\delta}}{\boldsymbol{\varphi}_2(1-\boldsymbol{\theta})(\boldsymbol{\rho}+\boldsymbol{\beta}_2)(\boldsymbol{\rho}+\boldsymbol{\alpha}_2)}\right] \quad (a145)$$

$$\text{Or, } \boldsymbol{b_i}^{so} = \frac{(1-\boldsymbol{\beta}_1)}{2\boldsymbol{\beta}_2}[\boldsymbol{\Omega}_2\boldsymbol{\gamma} + \boldsymbol{\Omega}_2\boldsymbol{\Omega}_3\boldsymbol{\delta}] \quad (a146)$$

Substituting eq. (a145) in eq. (a130),

$$m_i(t) = \frac{(1-\alpha_1)\delta\tau}{2\varphi_1\alpha_2(\rho+\alpha_2)} + \frac{\alpha_3}{\alpha_2}b_i(t) \quad (a147)$$

$$m_i(t) = \frac{(1-\alpha_1)\delta\tau}{2\varphi_1\alpha_2(\rho+\alpha_2)} + \frac{\alpha_3}{\alpha_2}\frac{(1-\beta_1)}{2\beta_2}\left[\frac{\gamma}{\varphi_2(\rho+\beta_2)} + \frac{\theta\alpha_3\delta}{\varphi_2(1-\theta)(\rho+\beta_2)(\rho+\alpha_2)}\right] \quad (a148)$$

$$\Rightarrow m_i^{so} = \frac{1}{2\alpha_2}\left[\frac{(1-\beta_1)\alpha_3}{\beta_2\varphi_2(\rho+\beta_2)}\right]\gamma + \frac{1}{2\alpha_2}\left[\frac{(1-\alpha_1)\tau}{\varphi_1(\rho+\alpha_2)} + \frac{(1-\beta_1)\theta\alpha_3^2}{\beta_2\varphi_2(1-\theta)(\rho+\beta_2)(\rho+\alpha_2)}\right]\delta \quad (a149)$$

$$\Rightarrow m_i^{so} = \frac{1}{2\alpha_2}\left[(1-\beta_1)\Omega_2\frac{\alpha_3}{\beta_2}\right]\gamma + \frac{1}{2\alpha_2}\left[(1-\alpha_1)\Omega_1 + (1-\beta_1)\Omega_2\Omega_3\frac{\alpha_3}{\beta_2}\right]\delta \quad (a150)$$

$$\Rightarrow \boldsymbol{m_i^{so}} = \frac{1}{2\boldsymbol{\alpha}_2}\left[\left((1-\boldsymbol{\beta}_1)\boldsymbol{\Omega}_2\frac{\boldsymbol{\alpha}_3}{\boldsymbol{\beta}_2}\right)\boldsymbol{\gamma} + \left((1-\boldsymbol{\alpha}_1)\boldsymbol{\Omega}_1 + (1-\boldsymbol{\beta}_1)\boldsymbol{\Omega}_2\boldsymbol{\Omega}_3\frac{\boldsymbol{\alpha}_3}{\boldsymbol{\beta}_2}\right)\boldsymbol{\delta}\right] \quad (a151)$$

This proves the proposition.

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Ganesh Manjhi, is Assistant Professor, Gargi College, Delhi University
Email: ganeshtrx@gmail.com

Meeta Keswani Mehra, Professor, Jawaharlal, Nehru University
Email: meetakm@mail.jnu.ac.in

National Institute of Public Finance and Policy,
18/2, Satsang Vihar Marg,
Special Institutional Area (Near JNU),
New Delhi 110067
Tel. No. 26569303, 26569780, 26569784
Fax: 91-11-26852548
www.nipfp.org.in